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Heat Losses in Conductors

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Heat Losses in Stranded Armature Conductors

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In the present paper, which is a continuation of one presented at the last annual convention, the author extends the method of complex hyperbolic functions to the solution of the problem of heat losses in stranded conductors embedded in rectangular slots. In the preceding paper the discussion was confined to solid conductors and to those having an infinite number of strands. In the latter case, the insulation between the strands was assumed to have no appreciable thickness. In the present paper, conductors are considered which have a finite number of strands separated by insulation of appreciable thickness. In the mathematical development which is to follow, free use is made of the results obtained in the first paper.

THE first step in the solution of the problem of heat losses in stranded armature conductors is to obtain an expression showing the relation between the currents in adjacent strands. If two strands are adjacent at any point, they are adjacent throughout their length. If the conductor is turned over in the end connection, the strand b, which is above a in one coil side, is below a in the next coil side, Fig. 1. The difference in pressure per half turn between these two strands is that between the lowest element of b and the highest element of a. If the conductor is turned over in the end connection, in the next half turn, this difference is between the lowest element of a and the highest element of b. The difference in pressure in each of these is cases given in equations (**6d**) and (**6r**). By the proper substitution, these equations can be written in terms of the currents in the adjacent strands and the current in the slot below them. See equations (**7a**) and (**7b**). These two equations are similar in form. Between the points at which the strands are joined together, the sum of all of these half-turn differences in pressure must be zero. This equation, (**8**), may be written in following form:

$$(I_{q(p+1)} - I_{qp}) - \frac{\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right)\frac{\alpha d}{m}}{\frac{\frac{\alpha d}{m}}{\sinh\frac{\alpha d}{m}} + \frac{l_2}{l_1}}\left(I_0 + \sum_{1}^{p} I_{qp}\right) = 0;$$

where α is calculated for the embedded portion of the winding. Compare this with the fundamental equation (**1**) in the preceding paper which for the case of infinitesimal strands may be written in the following form:

$$w\,dx\,d\underset{\cdot}{c} - \frac{\alpha^2 (dx)^2}{1 + \frac{l_2}{l_1}}\left(I_0 + \int_0^x w\,\underset{\cdot}{c}\,d\,x\right) = 0;$$

where α is also calculated for the embedded portion of the winding only, and l_1 and l_2 are respectively the length of the armature core and the length of an end turn. Notice that these equations are similar. The first term in each is the vector difference between the currents in adjacent strands. The coefficients of the second terms are the same when the number of strands, m, increases without limit and there is no insulating space, a, between them. It is further shown that the vector constants, I_0, are identical in the two equations. This similarity suggests that the currents in the strands of the stranded conductor may be equal to those in corresponding divisions of a solid conductor of the same depth. In the latter case, the imaginary strands would of course not be separated. (Fig. 2.)

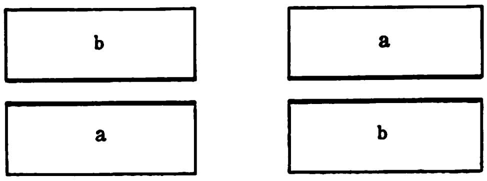

FIG. 1

Such proves to be the case provided the hyperbolic angular depth, $\beta\,d$, of the solid conductor has a certain value. This value is determined by the relation (Equation **10a**):

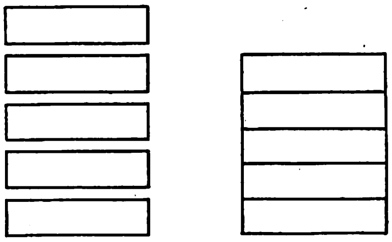

FIG. 2

$$\sinh^2\frac{\beta\,d}{2m} = \frac{\frac{\alpha\,d}{2\,m}\left(\frac{\alpha\,a}{2} + \tanh\frac{\alpha\,d}{2\,m}\right)}{\frac{\frac{\alpha\,d}{m}}{\sinh\frac{\alpha\,d}{m}} + \frac{l_2}{l_1}}$$

Were it not for this relation between the currents in the two cases the completion of the solution would be much more difficult. The reason is that the current

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up to any strand in the stranded conductor must be determined by a summation of vector quantities whereas in a solid conductor the current up to any point may be obtained by integration, a much simpler process. The current in any strand may also be obtained by integration between the proper limits.

This equality of the currents in a stranded conductor and a solid one recalls the equality of currents and of voltages that exists in an artificial transmission line and a smooth line when there is a certain relation between their constants.

We are now in a position to determine the expressions for the resistance drop in either the top or bottom strand of a conductor. It is only in the case of strands of infinitesimal thickness that the expressions are the same. Compare equations (**13b**) and (**14b**) with equation (**7**) in the preceding paper. They are identical except that M and N in the preceding paper are replaced by similar functions, M' and N' and that there are terms involving T' and S' which would disappear if the number of strands, m, were infinite and there were no insulating space, a, between them. We must also derive expressions for the pressure acting in conductors below a given one due to flux within the latter. These expressions are given in equations (**15b**) and (**16b**). These expressions are identical with the corresponding ones for solid or finely laminated conductors except that N' replaces N, that T' replaces $\alpha^2 d^2$ to which it reduces if there are an infinite number of laminations with no space between them, and that there is an added term involving S', a pure imaginary, which depends upon the insulating space between the laminations.

Having determined these four pressure equations, the determination of the heating loss or of the leakage impedance follows precisely the same procedure that was used in the preceding paper.

The author wishes to point out again a fact which he believes is not generally appreciated. A finely stranded conductor with the strands continuous throughout the whole coil will have a resistance ratio equal to that of a single finely stranded half turn of one half its depth, provided the coil has an even number of turns and the end connections are turned over on *one side only*. For conductors up to about one inch in depth, the alternating-current resistance is then but 1 or 2 per cent greater than the direct-current resistance.

Mathematical Analysis

The description of windings considered follows. The embedded portions of the winding are in open reactangular slots; there are two coil sides per slot; the pitch is either full or fractional; each coil side has n layers or conductors, and each layer has m laminations or strands; the strands occupy the same relative positions throughout the embedded portion of any half turn; the strands are arranged one above the other and have equal rectangular cross-sections; the strands are separated by insulation of uniform thickness; when the conductors are turned over or twisted in the end connections it is done in such a manner that any strand occupies the same relative position with respect to the bottom of the conductor of which it is a part that it does with respect to the top of the successive half turn; the coil as a whole is not turned over in the end connection; strands of any conductor may be joined at the beginning and end of a half turn, a full turn, or of a complete coil; the end connections may be turned over on neither side, on one side, or on both sides.

The strands are designated by the subscripts $q\,p$. The $q\,p$ strand is the p strand of the q conductor. The number of the strand may be counted from the bottom of the conductor of which it is a part, in which case the order of the strands is said to be direct, or from the top of the conductor of which it is a part, in which case the order of the strands is said to be reverse. Thus in a winding whose strands are continuous throughout a whole coil and whose end connnections are turned over on both sides the strands are in the direct order in one coil side and in the reverse order in the other coil side of the same coil. The number of the conductor is always counted from the bottom of the coil side of which it is a part.

The armature currents are assumed to be balanced. In the end connections the r. m. s. current density throughout any strand is assumed to be constant. In the embedded portion of the coil the r. m. s. current density in any strand is variable. This vector r. m. s. current density in the embedded portion of the $q\,p$ strand is:

$$\underset{\cdot}{c}_{q_p} = \frac{m}{w\,d}\left(I_{q_p}\,\frac{\alpha\,d}{m}\,\frac{\cosh\alpha\,x}{\sinh\dfrac{\alpha\,d}{m}} - I_{b\,q_p}\,\frac{\alpha\,d}{m}\tanh\frac{\alpha\,d}{2m}\cosh\alpha\,x + I_{b\,q_p}\,\frac{\alpha\,d}{m}\sinh\alpha\,x \right)^{1} \qquad (1)$$

$\underset{\cdot}{c}$ is the vector r. m. s. current density at points x centimeters from the bottom of the $p\,q$ strand; d is the net depth[2] of the conductor; d/m is the depth of one strand; w is the width of the conductor or strand; I_{q_p} is the current in the $q\,p$ strand; $I_{b\,q_p}$ is *all* of the current in the slot *below* the $q\,p$ strand;

$$\alpha^2 = j\,\frac{8\,\pi^2\,w\,f}{\rho\,s}\;;$$

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[1] See: Heat Losses in the Conductors of Alternating-Current Machines presented at Annual Convention A. I. E. E., June 1921. (Equation (**4**). For solid conductors like the $q\,p$ strand I_o is all of the current in the slot below the conductor considered.)

[2] The net depth of the conductor is the depth it would have if the insulation between the strands were of zero thickness.

where f is the frequency, w, the width of the conductor, and s, that of the slot, and ρ is the resistivity of the material of the conductor at the working temperature. ρ is assumed the same for every strand.

The current densities at the bottom, center and top of the $q\,p$ strand are respectively $c_{q_p b}$, $c_{q_p c}$ and $c_{q_p t}$. Substitution in equation (1) shows that
Since $x = O$,

$$c_{q_p b} = \frac{m}{w\,d}\left(I_{q_p} \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} - I_{b\,q_p} \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} \right) \qquad (2)$$

$$\text{Since } x = \frac{d}{2\,m},\ c_{q_p c} = \frac{m}{w\,d}\, I_{q_p} \frac{\frac{\alpha\,d}{2\,m}}{\sinh \frac{\alpha\,d}{2\,m}} \qquad (3)$$

$$\text{Since } x = \frac{d}{m},\ c_{q_p t} = \frac{m}{w\,d}\left(I_{q_p} \frac{\alpha\,d}{m} \coth \frac{\alpha\,d}{m} + I_{b\,q_p} \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} \right) \qquad (4)$$

or

$$c_{q_p t} = \frac{m}{w\,d}\left\{ I_{q_p} \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + I_{b\,q\,(p+1)} \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} \right\} \qquad (4a)$$

It is shown in the paper already referred to that the flux, ϕ_{q_p}, within the embedded portion of any solid conductor, the $_{q_p}$ strand for example, due to current within the conductor, I_{q_p} or example, and to all of the current in the slot below it, which in this case is $I_{b\,q_p}$, is

$$\phi_{q_p} = \frac{1}{j\,\omega}\,\frac{l_1\,\rho\,m}{w\,d}\left(I_{q_p} + 2 I_{b\,q_p} \right) \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} \qquad (5)$$

where l_1 is the length of the armature core. The flux, ϕ_a, between the $q\,(p+1)$ and the $q\,p$ strands is

$$\frac{4\,\pi\,l_1\,a}{s}\, I_{b\,q\,(p+1)},$$

which may be written in a form similar to equation (5). a is the thickness of the insulation between the strands

$$\phi_a = \frac{1}{j\,\omega}\,\frac{l_1\,\rho\,m}{w\,d}\left(I_{q_p} + I_{b\,q_p} \right) \frac{\alpha\,d}{m} \,.\, \alpha\,a \qquad (5a)$$

The proper combination of these five equations, *viz.*, (2), (3), (4), (5) and (5a) will give the heat loss in any of the types of stranded conductors we are considering. The leakage reactance due to flux within the embedded portion of the conductor may also be found.

Relation between the Currents in Adjacent Strands of the Same Conductor

For strands numbered in the direct order, the difference in pressure between adjacent elements of adjacent strands is the difference in pressure between that in the topmost element of the p strand and the bottom element of the $(p + 1)$ strand of the same conductor. For strands numbered in the reverse order this same difference is that between the pressures in the lowest element of the p strand and in the topmost element of the $(p + 1)$ strand. The total difference in pressure between these adjacent elements must be zero between the points at which the strands are joined.

For strands numbered in the direct order this difference in pressure, D_d, for a half turn is

$$D_d = \left(l_1\,\rho\,c_{q\,(p+1)\,b} + l_2 \frac{\rho\,m}{w\,d}\, I_{q\,(p+1)} \right) - \left(l_1\,\rho\,c_{q_p t} + l_2\,\rho \frac{m}{w\,d}\, I_{q_p} + l_1 \frac{\rho\,m}{w\,d}\, I_{b\,q\,(p+1)} \frac{\alpha\,d}{m} \,.\, \alpha\,a \right) \qquad (6d)$$

l_1 and l_2 are respectively the lengths of the armature core and of the end connections for a half turn. The second term in each parenthesis is the resistance drop in the end connections. The third term in the second parenthesis is the drop in pressure in the $q\,p$ strand due to the flux ϕ_a. The reactance drop in the end connections due to internal leakage flux is neglected, *i. e.*, the current density is assumed to be uniform in each strand of the end connections.

Substitute equations (2) and (4) in equation (6d)

$$D_d = \frac{m}{w\,d}\,\rho \left\{ \left[l_1\, I_{q\,(p+1)} \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} - l_1\, I_{b\,q\,(p+1)} \frac{\alpha\,d}{m}\left(\alpha\,a + \tanh \frac{\alpha\,d}{2\,m} \right) + l_2\, I_{q\,(p+1)} \right] - \left[l_1\, I_{q_p} \frac{\alpha\,d}{m} \coth \frac{\alpha\,d}{m} + l_1\, I_{b\,q_p} \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} + l_2\, I_{q_p} \right] \right\}$$

but $\qquad I_{b\,q_p} = I_{b\,q\,(p+1)} - I_{q_p}$
Thus:

$$D_d = \frac{m}{w\,d}\,\rho \left\{ \left(l_1 \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + l_2 \right) \left(I_{q\,(p+1)} - I_{q_p} \right) - l_1\, I_{b\,q\,(p+1)} \frac{\alpha\,d}{m}\left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \right\} \qquad (6a)$$

For strands numbered in the reverse order the pressure difference, D_r, for a half turn is:

$$D_r = \left(l_1\,\rho\,c_{q\,(p+1)\,t} + l_2\,\rho \frac{m}{w\,d}\, I_{q\,(p+1)} \right.$$

$$+ l_1 \rho \frac{m}{w\,d}\; I_{b q_p} \frac{\alpha\,d}{m} \,.\, \alpha\,a \Big)$$

$$- \left(l_1 \rho\, c_{q_p b} + l_2 \rho \frac{m}{w\,d} I_{q_p} \right) \qquad (\mathbf{6r})$$

Substitute as before

$$D_r = \frac{m}{w\,d} \rho \left\{ \left[l_1 I_{q\,(p+1)} \frac{\alpha\,d}{m} \coth \frac{\alpha\,d}{m} \right.\right.$$

$$\left. + l_1 I_{bq\,(p+1)} \frac{\alpha\,d}{m} \tanh \frac{\alpha\,d}{2\,m} + l_2 I_{q\,(p+1)} \right]$$

$$- \Bigg[l_1 I_{q_p} \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}}$$

$$\left.\left. - l_1 I_{b q_p} \frac{\alpha\,d}{m} \left(\alpha\,a + \tanh \frac{\alpha\,d}{2\,m} \right) + l_2 I_{q_p} \right] \right\}$$

Now $\qquad I_{b q_p} = I_{bq\,(p+1)} + I_{q\,(p+1)}$

Thus:

$$D_r = \frac{m}{w\,d} \rho \left\{ \left(l_1 \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + l_2 \right) \left(I_{q\,(p+1)} - I_{q_p} \right) \right.$$

$$\left. + l_1 I_{b q_p} \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \right\} \qquad (\mathbf{6b})$$

In equation (**6a**) $\qquad I_{bq\,(p+1)} = I_{bq1} + \sum_1^p I_{q_p};$ where the first part is the current in the slot below the q conductor and the second part is the current in the q conductor below its $(p + 1)$ strand. In equation (**6b**) $I_{b q_p} = I_{bqm} + I_q - \sum_1^p I_{q_p}$ where the first part is the current in the slot below the q conductor, the second part is the current in the q conductor and the third part, as before, is the current in the q conductor from the first strand to the p strand inclusive.

With these substitutions, equations (**6a**) and (**6b**) become

$$D_d = \frac{m\,\rho}{w\,d} \left\{ \left(l_1 \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + l_2 \right) \left(I_{q\,(p+1)} - I_{q_p} \right) \right.$$

$$- l_1 \frac{\alpha\,d}{2\,m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) I_{bq1}$$

$$- l_1 \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \sum_1^p I_{q_p} \qquad (\mathbf{7a})$$

$$D_r = \frac{m\,\rho}{w\,d} \left\{ \left(l_1 \frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + l_2 \right) \left(I_{q\,(p+1)} - I_{q_p} \right) \right.$$

$$+ l_1 \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \left(I_{bqm} + I_q \right)$$

$$\left. - l_1 \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \sum_1^p I_{q_p} \right\} \qquad (\mathbf{7b})$$

Between the points at which the strands are joined together the sum of all of these differences in pressure in adjacent half-turn elements, *viz.*, $\Sigma\,(D_d + D_r)$, must be zero. In general this sum may be written in the form

$$\left(\frac{\frac{\alpha\,d}{m}}{\sinh \frac{\alpha\,d}{m}} + \frac{l_2}{l_1} \right) \left(I_{q\,(p+1)} - I_{q_p} \right)$$

$$- \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) I_0$$

$$- \frac{\alpha\,d}{m} \left(\alpha\,a + 2 \tanh \frac{\alpha\,d}{2\,m} \right) \sum_1^p I_{q_p} = 0 \qquad (\mathbf{8})$$

where I_0 is the sum of the I_{bq1}'s and the $(-I_{bqm} - I_q)$'s divided by $2\,n$, that is in general

$$I_0 = \frac{1}{2\,n} \left(I_{bq1} - I_{bqm} - I_q \right)$$

We will now calculate the values of I_0 for the cases that we wish to consider.

Case 1. *Strands joined at the beginning and end of each half turn.* For the q conductor of the upper coil side

$$I_0 = (q - 1)\,I + n\,I \underline{/\theta}$$

The first part of I_0 is the current in the upper coil side below the q conductor and the second part is the current in the lower coil side which in general differs in phase by an angle θ. There are n conductors in each coil side. The current in each conductor is I amperes $(I_q = I)$.

Case 2. *Strands joined at the beginning and end of a whole turn but not turned over in the end connection.* For the q conductor of the coil:

$$I_0 = 1/2 \left\{ (q - 1)\,I + n\,I \underline{/\theta} + (q - 1)\,I \right\}$$
$$= (q - 1)\,I + n/2\,I \underline{/\theta}$$

Case 3. *Strands joined at the beginning and end of a whole turn and turned over in the end connection.* For the q conductor:

$$I_0 = 1/2 \left\{ (q - 1)\,I + n\,I \underline{/\theta} - (q - 1)\,I - I \right\}$$

$$I_0 = -\,I/2 + \frac{n\,I \underline{/\theta}}{2}$$

Case 4. *Strands joined at the beginning and end of a whole coil of n turns. End connection not turned over on either side.*

$$I_0 = \frac{1}{2\,n} \sum_1^n \left[(q - 1)\,I + n\,I \underline{/\theta} + (q - 1)\,I) \right]$$

$$= \frac{n - 1}{2}\,I + n/2\,I \underline{/\theta}$$

Case 5. *Strands joined at the beginning and end of a whole coil of n turns. End connections turned over on one side only.* The order of the strands in the first, or lowest, conductor of the upper coil side is direct, that of the next half turn is reverse. The strands in the second conductor of the upper coil side are in the reverse order while those in the next half turn are in the direct order.

In general

$$I_0 = \frac{1}{2n}\Big\{ (n\,\underline{/\theta} - 1) + (-n\,\underline{/\theta} - 1 - 1 + 1) + (n\,\underline{/\theta} + 2 - 2 - 1) + (-n\,\underline{/\theta} - 3 - 1 + 3) + \text{etc.}\Big\}\, I$$

$$I_0 = \frac{1}{2n}\Big\{ (n\,\underline{/\theta} - 1) + (-n\,\underline{/\theta} - 1) + (n\,\underline{/\theta} - 1) + (-n\,\underline{/\theta} - 1)\ \text{etc.}\Big\}\, I$$

(a) n an even number

$I_0 = -I/2$

(b) n an odd number

$I_0 = -I/2 + I/2\,\underline{/\theta}$

Case 6. *Strands joined at the beginning and end of a whole coil of n turns. End connections turned over on both sides.* In the upper coil side the strands are all in the direct order while in the lower coil side they are all in the reverse order.

$$I_0 = \frac{1}{2n}\sum_1^n (n\,\underline{/\theta} + (q-1) - (q-1) - 1)\, I = -I/2 + \frac{n\, I\,\underline{/\theta}}{2}$$

Consider a solid conductor of the same width, w, and same net depth, d, but which has an angle β instead of α. The current density in this conductor is

$$c = A' \cosh \beta x + B' \sinh \beta x$$

Imagine that this solid conductor is divided into m equal parts, *i. e.*, strands, in the same manner that the actual conductor is divided. The difference between the currents in the $(p+1)$ and p strands is

$$I'_{p+1} - I_p' = \frac{w\,d}{m}\,\frac{\sinh\frac{\beta d}{2m}}{\frac{\beta d}{2m}}\left(c'_{(p+1)c} - c_{pc}'\right)$$

(See equation 2)

But
$$c'_{(p+1)c} - c'_{pc} = \left(A' \cosh\frac{p + 1/2}{m}\,\beta d + B' \sinh\frac{p+1/2}{m}\,\beta d\right) - \left(A' \cosh\frac{p - 1/2}{m}\,\beta d + B' \sinh\frac{p-1/2}{m}\,\beta d\right)$$

Expand in terms of $p/m\ \beta d$ and $\frac{\beta d}{2m}$ giving

$$c'_{(p+1)c} - c'_{pc} = (A' \sinh p/m\ \beta d + B' \cosh p/m\ \beta d)\ 2 \sinh\frac{\beta d}{2m}$$

Thus:

$$I'_{p+1} - I'_p = \frac{w\,d}{m}\,\frac{2\sinh^2\frac{\beta d}{2m}}{\frac{\beta d}{2m}}\,(A' \sinh p/m\ \beta d + B' \cosh p/m\ \beta d)$$

The current in this conductor up to and including the p part is

$$\sum_1^p I'_p = \int_0^{\frac{p}{m}d} w\,c'\,dx = w/\beta\,(A' \sinh p/m\ \beta d + B' \cosh p/m\ \beta d) - w/\beta\, B'$$

We will now show that these values of current can be made to satisfy the relation that has been established between the currents in adjacent strands of the actual conductor. Substitute these values of $(I'_{p+1} - I'_p)$ and $\sum_1^p I_p'$ in equation (8). That is assume that the currents in the imaginary strands of the solid conductor are exactly equal to the currents in the corresponding strands of the actual conductor

$$\left(\frac{\frac{\alpha d}{m}}{\sinh\frac{\alpha d}{m}} + \frac{l_2}{l_1}\right)\frac{w\,d}{m}\,\frac{2\sinh^2\frac{\beta d}{2m}}{\frac{\beta d}{2m}}\,(A' \sinh p/m\ \beta d + B' \cosh p/m\ \beta d) - \frac{\alpha d}{m}\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right) I_0 - \frac{\alpha d}{m}\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right) w/\beta\left(A' \sinh p/m\,\beta d + B' \cosh p/m\,\beta d - B'\right) = 0 \qquad \textbf{(8a)}$$

This equation (**8a**) is satisfied if:

$$\left(\frac{\frac{\alpha d}{m}}{\sinh\frac{\alpha d}{m}} + \frac{l_2}{l_1}\right)\frac{w\,d}{m}\,\frac{2\sinh^2\frac{\beta d}{2m}}{\frac{\beta d}{2m}} - \frac{\alpha d}{m}\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right) w/\beta = 0 \qquad \textbf{(9a)}$$

and:

$$-\frac{\alpha d}{m}\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right) I_0 + \frac{\alpha d}{m}\left(\alpha a + 2\tanh\frac{\alpha d}{2m}\right) w/\beta\, B' = 0 \qquad \textbf{(9b)}$$

These conditions readily reduce to

$$\sinh^2 \frac{\beta d}{2m} = \frac{\dfrac{\alpha d}{2m}\left(\dfrac{\alpha a}{2} + \tanh \dfrac{\alpha d}{2m}\right)}{\dfrac{\dfrac{\alpha d}{m}}{\sinh \dfrac{\alpha d}{m}} + \dfrac{l_2}{l_1}} \qquad \textbf{(10a)}$$

and $$B' = \beta / w \, I_0 \qquad \textbf{(10b)}$$

If the total current in the conductor, actual or equivalent, is I, the vector constant A' is determined as in the preceding paper.

$$A' = \beta / w \left(\frac{I}{\sinh \beta d} - I_0 \tanh \frac{\beta d}{2} \right) \qquad \textbf{(10c)}$$

By equation (3) the current in the p strand is

$$I_p = \frac{w\,d}{m} \, \frac{\sinh \dfrac{\beta d}{2m}}{\dfrac{\beta d}{2m}} \, c_{pc}$$

$$= \frac{w\,d}{m} \, \frac{\sinh \dfrac{\beta d}{2m}}{\dfrac{\beta d}{2m}}$$

$$\left(A' \cosh \frac{p - 1/2}{m} \, \beta d + B' \sinh \frac{p - 1/2}{m} \, \beta d \right)$$

$$= 2 \sinh \frac{\beta d}{2m} \left(I \, \frac{\cosh \dfrac{p - 1/2}{m} \, \beta d}{\sinh \beta d} \right.$$

$$- I_0 \tanh \frac{\beta d}{2} \cosh \frac{p - 1/2}{m} \, \beta d$$

$$\left. + I_0 \sinh \frac{p - 1/2}{m} \, \beta d \right) \qquad \textbf{(11)}$$

The current in the conductor below the p strand is

$$\sum_{1}^{p-1} I_p = \int_{0}^{\frac{p-1}{m} d} w \, c' \, d\,x$$

$$= w / \beta \left(A' \sinh \frac{p-1}{m} \, \beta d + B' \cosh \frac{p-1}{m} \, \beta d - B_1 \right)$$

$$= \frac{\sinh \dfrac{p-1}{m} \, \beta d}{\sinh \beta d}$$

$$- I_0 \tanh \frac{\beta d}{2} \sinh \frac{p-1}{m} \, \beta d$$

$$+ I_0 \cosh \frac{p-1}{m} \, \beta d - I_0 \qquad \textbf{(12)}$$

The total current below the $q\,p$ strand of the actual conductor is

$$I_{bq} + \sum_{1}^{p-1} I_{qp},$$

where I_{bq} is the current in the slot below the q conductor.

If the current in any conductor and all of that below it in the slot are given the copper loss in the conductor may be calculated as described in the preceding paper. We are thus able to calculate the copper loss in any strand of any conductor. This method of calculation is far too laborious and we shall content ourselves with calculating the loss for a half turn (Case 1), a single turn (Cases 2 and 3) or for a single coil (Cases 4, 5 and 6). To do this it is necessary to obtain expressions for the voltage drop per half turn in the top element[3] of the top strand, in the bottom element[3] of the bottom strand and the voltage drop due to flux within a single conductor in all conductors below it.

The voltage drop per half turn in the topmost element of the $q\,p$ strand due to resistance and leakage below the $q\,(p + 1)$ element is,

$$l_1 \, \rho \, c_{qpt} + l_2 \, \rho \, \frac{m}{w\,d} \, I_{qp} + l_1 \, \rho \, \frac{m}{w\,d} \, I_{bq\,(p+1)} \, \frac{\alpha d}{m} \, \alpha a$$

The third term in this expression is the voltage drop due to flux within the insulation—of thickness a—that is immediately above the $q\,p$ strand. It is necessary to include this term in order that the voltage expressions about to be derived will be similar in certain respects to the general equation (8) for the currents in the strands.

By equation (4a) this reduces to

$$\rho \, \frac{m}{w\,d} \left\{ \left(l_1 \, \frac{\dfrac{\alpha d}{m}}{\sinh \dfrac{\alpha d}{m}} + l_2 \right) I_{qp} \right.$$

$$\left. + \, l_1 \, I_{bq\,(p+1)} \, \frac{\alpha d}{m} \left(\alpha a + \tanh \frac{\alpha d}{2m} \right) \right\}$$

The voltage drop per half turn in the topmost element of the top strand of the q conductor of the upper coil side due to resistance and leakage flux below this element is,

$$\text{drop} = \rho \, \frac{m}{w\,d} \, l_1 \left[\left(\frac{\dfrac{\alpha d}{m}}{\sinh \dfrac{\alpha d}{m}} + \frac{l_2}{l_1} \right) I_{qm} \right.$$

$$\left. + \, (I_{bq1} + I) \, \frac{\alpha d}{m} \left(\alpha a + \tanh \frac{\alpha d}{2m} \right) \right] \qquad \textbf{(13)}$$

This pressure is used when the strands are numbered in the direct order. The loss in pressure per half turn in the lowest element of the bottom strand of the $q\,p$ conductor due to resistance and leakage flux below this element is:

$$l_1 \, \rho \, c_{qpb} + l_2 \, \rho \, \frac{m}{w\,d} \, I_{qp}$$

---

[3] Due to resistance and flux below it.

By equation (2) this reduces to:

$$\text{drop} = \rho \frac{m}{w\,d}\, l_1 \left\{ \left(\frac{\frac{\alpha\, d}{m}}{\sinh \frac{\alpha\, d}{m}} + \frac{l_2}{l_1} \right) I_{q_p} - I_{b\,q_p} \frac{\alpha\, d}{m} \tanh \frac{\alpha\, d}{2\, m} \right\}$$

The loss in pressure per half turn in the lowest element of the bottom—*i. e.* the q_1—strand of the q conductor of the upper coil side due to resistance and leakage flux below this element is,

$$\text{drop} = \rho \frac{m}{w\,d}\, l_1 \left\{ \left(\frac{\frac{\alpha\, d}{m}}{\sinh \frac{\alpha\, d}{m}} + \frac{l_2}{l_1} \right) I_{q1} - I_{b\,q1} \frac{\alpha\, d}{m} \left(\alpha\, a + \tanh \frac{\alpha\, d}{2\, m} \right) + I_{b\,q1} \frac{\alpha\, d}{m} \,.\, \alpha\, a \right\} \tag{14}$$

This pressure is used when the strands are numbered in the reverse order. The equation is written in this form so that it will be similar to equation (13). When the strands are continuous from one half turn to the next and the end connection is turned over between successive half turns the top strand of one half turn becomes the bottom strand of the next half turn. Thus in equations (13) and (14) the I_{q_m} is the same as I_{q1}.

The flux in the q conductor is,

$$\phi_q = \sum_{1}^{m} (\phi_{q_p} + \phi_a)$$

By equations (5) and (5a) this is,

$$\begin{aligned} \phi_q &= \frac{1}{j\,\omega}\, l_1\, \rho \frac{m}{w\,d} \frac{\alpha\, d}{m} \left\{ \tanh \frac{\alpha\, d}{2\, m} \sum_1^m (I_{q_p} + 2\, I_{b\,q_p}) + \alpha\, a \sum_1^m (I_{q_p} + I_{b\,q_p}) \right\} \\ &= \frac{1}{j\,\omega}\, \rho \frac{m}{w\,d}\, l_1 \left\{ I \frac{\alpha\, d}{m} \left(\alpha\, a + \tanh \frac{\alpha\, d}{2\, m} \right) + \sum_1^m I_{b\,q_p} \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) \right\} \end{aligned}$$

The summation in the second term in this expression for the flux may be written,

$$\begin{aligned} \sum_1^m I_{b\,q_p} = \Big(& I_{b\,q1} + 0 \\ & + I_{b\,q1} + I_{q1} \\ & + I_{b\,q1} + I_{q1} + I_{q2} \\ & + I_{b\,q1} + I_{q1} + I_{q2} + I_{q3} \\ & \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + I_{b\,q1} + \sum_1^{p-1} I_{q_p} \\ & \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + I_{b\,q1} + \sum_1^{m-1} I_{q_p} \\ = {} & m\, I_{b\,q1} + \sum_1^{m-1} \sum_1^{p-1} I_{q_p} \Big) \end{aligned}$$

From equation (8) we have,

$$\begin{aligned} \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) \sum_1^{p-1} I_{q_p} &= \left(\frac{\frac{\alpha\, d}{m}}{\sinh \frac{\alpha\, d}{m}} + \frac{l_2}{l_1} \right) (I_{q_p} - I_{q\,(p-1)} \\ & \quad - \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) I_0 \end{aligned}$$

If the $(m - 1)$ equations for the adjacent strands are added together we have,

$$\begin{aligned} \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) \sum_1^{m-1} \sum_1^{p-1} I_{q_p} &= \left(\frac{\frac{\alpha\, d}{m}}{\sinh \frac{\alpha\, d}{m}} + \frac{l_2}{l_1} \right) (I_{q_m} - I_{q1}) \\ & \quad - (m - 1) \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{m} \right) I_0 \end{aligned}$$

Therefore the flux in the q conductor when the strands are numbered in the direct order is,

$$\begin{aligned} \phi_q = \frac{1}{j\,\omega}\, \rho \frac{m}{w\,d}\, l_1 \Bigg[& I \frac{\alpha\, d}{m} \left(\alpha\, a + \tanh \frac{\alpha\, d}{2\, m} \right) \\ & + \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) m\, I_{b\,q1} \\ & + \left(\frac{\frac{\alpha\, d}{m}}{\sinh \frac{\alpha\, d}{m}} + \frac{l_2}{l_1} \right) (I_{q_m} - I_{q1}) \\ & - (m - 1) \frac{\alpha\, d}{m} \left(\alpha\, a + 2 \tanh \frac{\alpha\, d}{2\, m} \right) I_0 \Bigg] \end{aligned} \tag{15}$$

If the strands are numbered in the reverse order, so that the first strand is at the top and the mth strand is at the bottom, the current in the conductor below the $q\,p$ strand is $I - \sum_1^p I_{q_p}$, and the $\sum_1^m I_{b\,q_p}$ is now:

$$\begin{aligned} \sum_1^m I_{b\,q_p} &= I_{b\,q_m} + I - I_{q1} \\ & \quad + I_{b\,q_m} + I - (I_{q1} + I_{q2}) \\ & \quad + I_{b\,q_m} + I - \sum_1^p I_{q_p} \\ & \quad + \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & \quad + I_{b\,q_m} + I - I \\ &= m\, (I_{b\,q_m} + I) - \sum_1^{m-1} \sum_1^{p} I_{q_p} - I \\ &= m\, I_{b\,q_m} + (m - 1)\, I - \sum_1^{m-1} \sum_1^{p} I_{q_p} \end{aligned}$$

Therefore the flux in the q conductor when the strands are numbered in the reverse order is

$$\phi_q = \frac{1}{j\,\omega}\,\rho\,\frac{m}{w\,d}\,l_1\Bigg[\,I\,\frac{\alpha\,d}{m}\left(\alpha\,a + \tanh\frac{\alpha\,d}{2\,m}\right)$$

$$+\,\frac{\alpha\,d}{m}\left(\alpha\,a + 2\tanh\frac{\alpha\,d}{2\,m}\right)(m\,I_{b\,q_m} + (m-1)\,I)$$

$$-\left(\frac{\dfrac{\alpha\,d}{m}}{\sinh\dfrac{\alpha\,d}{m}} + \frac{l_2}{l_1}\right)(I_{q_m} - I_{q_1})$$

$$+\,(m-1)\,\frac{\alpha\,d}{m}\left(\alpha\,a + 2\tanh\frac{\alpha\,d}{2\,m}\right)I_0\Bigg] \qquad \textbf{(16)}$$

The loss in pressure as derived in equations (**13**) and (**14**) and the flux within a conductor depend upon the current in the top strand and the difference between the currents in the top and bottom strands. The current in the p strand is determined by equation (**11**).

The current in the top strand numbered in direct order, is I_{q_m}.

$$I_{q_m} = 2\sinh\frac{\beta\,d}{2\,m}\Bigg(I\,\frac{\cosh\dfrac{m-1/2}{m}\,\beta\,d}{\sinh\,\beta\,d}$$

$$-\,I_0\tanh\frac{\beta\,d}{2}\cosh\frac{m-1/2}{m}\,\beta\,d$$

$$+\,I_0\sinh\frac{m-1/2}{m}\,\beta\,d\Bigg)$$

Expand in terms of $\beta\,d$ and $\dfrac{\beta\,d}{2\,m}$

$$I_{q_m} = 2\sinh^2\frac{\beta\,d}{2\,m}\Bigg[\,I\left(\coth\,\beta\,d\coth\frac{\beta\,d}{2\,m} - 1\right)$$

$$+\,I_0\left(\tanh\frac{\beta\,d}{2}\coth\frac{\beta\,d}{2\,m} - 1\right)\Bigg] \qquad \textbf{(17)}$$

Likewise:

$$I_{q_1} = 2\sinh^2\frac{\beta\,d}{2\,m}\Bigg[\,I\,\frac{\coth\dfrac{\beta\,d}{2\,m}}{\sinh\,\beta\,d}$$

$$-\,I_0\left(\tanh\frac{\beta\,d}{2}\coth\frac{\beta\,d}{2\,m} - 1\right)\Bigg]$$

Thus:

$$(I_{q_m} - I_{q_1}) = 2\sinh^2\frac{\beta\,d}{2\,m}\Bigg[\,(I + 2\,I_0)$$

$$\left(\tanh\frac{\beta\,d}{2}\coth\frac{\beta\,d}{2\,m} - 1\right)\Bigg] \qquad \textbf{(18)}$$

The pressure equations which determine the resistance and reactance drops, *viz.* equations (**13**) to (**16**) inclusive are in terms of these currents in the top and bottom strands in addition to the conductor current and the current I_0. We will rewrite these equations, making the substitutions given in equations (**17**) and (**18**), and remembering that the I_{q_m} in (**17**) is the same as I_{q_1} in (**14**). We will also make the substitution shown in equation (**10a**).

Equation (**13**) becomes:

$$\text{drop} = \frac{\rho}{w\,d}\,l_1\Bigg[\,I\,\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta\,d}{2\,m}}\coth\,\beta\,d$$

$$+\,I_0\,\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta d}{2\,m}}\tanh\frac{\beta\,d}{2}$$

$$+\,(I_b - I_0)\,\alpha\,d\left(\frac{\alpha\,a}{2} + \tanh\frac{\alpha\,d}{2\,m}\right)$$

$$+\,(I + I_b)\,\alpha\,d\,.\,\frac{\alpha\,a}{2}\Bigg] \qquad \textbf{(13a)}$$

Equation (**14**) becomes:

$$\text{drop} = \frac{\rho}{w\,d}\,l_1\Bigg[\,I\,\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta\,d}{2\,m}}\coth\,\beta\,d$$

$$+\,I_0\,\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta\,d}{2\,m}}\tanh\frac{\beta\,d}{2}$$

$$-\,(I_b + I_0 + I)\,\alpha\,d\left(\frac{\alpha\,a}{2} + \tanh\frac{\alpha\,d}{2\,m}\right)$$

$$+\,I_b\,\alpha\,d\,.\,\frac{\alpha\,a}{2}\Bigg] \qquad \textbf{(14a)}$$

Equation (**15**) becomes:

$$\phi_q = \frac{1}{j\,\omega}\,\frac{\rho}{w\,d}\,l_1\Bigg[\,(I + 2\,I_0)$$

$$\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta\,d}{2\,m}}\tanh\frac{\beta\,d}{2}$$

$$+\,(I_b - I_0)\,2\,m\,\alpha\,d\left(\frac{\alpha\,a}{2} + \tanh\frac{\alpha\,d}{2\,m}\right)$$

$$+\,I\,\alpha\,d\,.\,\frac{\alpha\,a}{2}\Bigg] \qquad \textbf{(15a)}$$

Equation (**16**) becomes:

$$\phi_q = \frac{1}{j\,\omega}\,\frac{\rho}{w\,d}\,l_1\Bigg[\,-\,(I + 2\,I_0)$$

$$\alpha\,d\,\frac{\left(\dfrac{\alpha\,a}{2} + \tanh\dfrac{\alpha\,d}{2\,m}\right)}{\tanh\dfrac{\beta\,d}{2\,m}}\tanh\frac{\beta\,d}{2}$$

$$+ (I_b + I_0 + I)\, \alpha\, d\, 2\, m \left(\frac{\alpha\, a}{2} + \tanh \frac{\alpha\, d}{2\, m} \right)$$

$$+ I\, \alpha\, d\, .\, \frac{\alpha\, a}{2} \Bigg] \qquad \textbf{(16a)}$$

In each of these four equations I_b is the current in the slot below the conductor in question.

The writing of these equations is much simplified if we let:

$$\frac{l_1}{l_1 + l_2}\, \alpha\, d\, \frac{\left(\frac{\alpha\, a}{2} + \tanh \frac{\alpha\, d}{2\, m} \right)}{\tanh \frac{\beta\, d}{2\, m}} \coth \beta\, d$$

$$= M' = M_r' + j\, M_x'$$

$$\frac{l_1}{l_1 + l_2}\, \alpha\, d\, \frac{\left(\frac{\alpha\, a}{2} + \tanh \frac{\alpha\, d}{2\, m} \right)}{\tanh \frac{\beta\, d}{2\, m}}\, 2 \tanh \frac{\beta\, d}{2}$$

$$= N' = N_r' + j\, N_x'$$

$$\frac{l_1}{l_1 + l_2}\, \alpha\, d\, 2\, m \left(\frac{\alpha\, a}{2} + \tanh \frac{\alpha\, d}{2\, m} \right)$$

$$= T' = T_r' + j\, T_x'$$

$$\frac{l_1}{l_1 + l_2}\, \alpha\, d\, \alpha\, a = S' = j\, \frac{8\, \pi^2\, \omega\, f}{\rho\, s}\, d\, a$$

and $\frac{\rho}{wd} (l_1 + l_2) = R$ (the d-c. resistance of a half turn)

These equations may now be rewritten in simplified form. The resistance drop per half turn in the top element of the top strand of a conductor (**13a**) becomes:

$$\text{drop} = R \left[I\, M' + \frac{I_0\, N'}{2} + (I_b - I_0)\, \frac{T'}{2\, m} \right.$$

$$\left. + (I + I_b)\, S'/2 \right] \qquad \textbf{(13b)}$$

If the strands are turned over in the end connections the resistance drop per half turn in the bottom element of the foregoing strand, which is now the lowest in the conductor (**14a**) becomes:

$$\text{drop} = R \left[I\, M' + \frac{I_0\, N'}{2} - (I_b + I_0 + I)\, \frac{T'}{2\, m} \right.$$

$$\left. + I_b\, S'/2 \right] \qquad \textbf{(14b)}$$

The average resistance drop per half turn for the six cases considered is obtained from these two equations. By definition I_0 has such a value that the term involving T' in this average resistance drop is zero in every case. In order to simplify the expression we will let I_s represent the coefficient of $S'/2$. The average resistance drop per half turn is:

$$\text{av. drop} = R \left(I\, M' + \frac{I_0\, N'}{2} + \frac{I_s\, S'}{2} \right) \qquad \textbf{(19)}$$

I_s is the average sum of the $(I + I_b)$'s in equation (**13a**) and the I_b's in equation (**14a**) taken in the proper combination.

Case 1. $I_s = I + (q - 1 + n \angle \theta)\, I$ upper coil side
$= (q + n \angle \theta)\, I$ upper coil side
$= q\, I$ lower coil side

Case 2. $I_s = 1/2 \{ [I + (q - 1 + n \angle \theta)\, I] + [I + (q - 1)\, I] \}$
$= (q + n/2 \angle \theta)\, I$

Case 3. $I_s = 1/2 \{ [I + (q - 1 + n \angle \theta)\, I] + [(q - 1)\, I] \}$
$= (q - 1/2 + n/2 \angle \theta)\, I$

Case 4. $I_s = 1/n \sum_1^n (q + n/2 \angle \theta)\, I$

$$= \left(\frac{n + 1}{2} + n/2 \angle \theta \right) I$$

Case 5. $I_s = \frac{1}{2\, n} \Big[(1 + n \angle \theta) + (O)$
$+ (1 + n \angle \theta) + (1 + 1)$
$+ (1 + 2 + n \angle \theta) + (2) + \text{etc.} \Big]\, I$

The first term within the parenthesis is the current in the conductor plus the current below the first conductor of the upper coil side (**13b**). The second term is the current below the first conductor of the lower coil side (**14b**). This is zero. The third term is the current below the second conductor of the upper coil side (**14b**). The fourth term is the current in the conductor plus the current below the second conductor of the lower coil side (**13b**). Notice that this expression for I_s may be written:

$$I_s = \frac{1}{2\, n} \Big[(1 + n \angle \theta) + (3 + n \angle \theta) + (5 + n \angle \theta) + \ldots . + (2\, n - 1 + n \angle \theta) \Big) I$$

$$I_s = \frac{1}{2\, n} \sum_1^n (2\, q - 1 + n \angle \theta)\, I$$

$$I_s = n/2 + n/2 \angle \theta)\, I$$

Case 6. $I_s = 1/n \sum_1^n (q - 1/2 + n/2 \angle \theta)\, I$
$= (n/2 + n/2 \angle \theta)\, I$

The pressure acting in all conductors below the q conductor due to flux within the latter is $j\, \omega\, \phi_q$. Written in the simplified form this (**15a**) becomes:

$$\text{drop} = R\, [\, (I/2 + I_0)\, N' + (I_b - I_0)\, T' + I/2\, S'] \qquad \textbf{(15b)}$$

If the strands are turned over in the end connections, the pressure due to flux within the next succeeding half turn (**16a**) becomes:

$$\text{drop} = R\, [- (I/2 + I_0)\, N' + (I_b + I_0 + I)\, T' + I/2\, S' \qquad \textbf{(16b)}$$

Reference to the preceding paper shows that these pressure expressions are similar to those already derived for solid and finely laminated conductors except for the added terms involving S'. M' replaces M, N' replaces N and T' replaces $\alpha^2 d^2$. The expressions become identical in the limiting case of an infinite number of strands with no insulation between them.

CALCULATION OF COPPER LOSS

The method of calculation is the same as that used in the preceding paper.

Case 1. When the strands are joined at the beginning and end of a half turn the copper loss in the half turn is symbolically:

$$\text{loss} = I \,.\, R\,(I\,M' + I_b/2\,N' + I_s\,2\,S') \\ + I_b \,.\, R\,[\,(I/2 + I_s/2)\,N' + I/2\,S']$$

In this case notice that $I_b = I_0$

The first term is the power loss due to the current in the conductor and the resistance drop in the topmost element of the topmost strand. The second term is the power due to the current in the slot below this conductor and to the pressure acting on it produced by the flux within the conductor. Expanding this expression gives:

$$\text{power} = I \,.\, R\,I\,M' + I \,.\, R\,\frac{I_b}{2}N' + I \,.\, R\,\frac{I_s}{2}\,S' \\ + I_b \,.\, R\,\frac{I}{2}N' + I_b \,.\, R\,I_b\,N' + I_b \,.\, R\,\frac{I}{2}\,S'.$$

The first term is $R\,I^2\,M_r'$. The sum of the second and fourth terms is $R\,I\,I_b\,N_r'\,\cos\theta_b$. The fifth term is $R\,I_b^2\,N_r'$ and since S' is a pure imaginary the sum of the third and sixth terms is zero. The expression for the power may be written:

$$\text{loss} = R\,[I^2\,M_r' + (I \,.\, I_b \cos\theta_b + I_b^2)\,N_r']$$

The phase angle θ_b is between the current, I, in the conductor and the total current, I_b, in the slot below it. Here the letters I and I_b represent the numerical values of the currents.

If this copper loss due to alternating current is divided by the loss due to the same amount of direct current, we obtain the ratio of alternating to direct-current resistance.

(a) The ratio of alternating to direct-current resistance for a single half turn is:

$$K = \{M_r' + [\,(I_b/I)^2 + I_b/I \cos\theta_b]\,N_r'\}$$

(b) The average resistance ratio for a one-coil-side-per-slot winding having n layers is:

$$K = 1/n \sum_1^n \{M_r' + [\,(q-1)^2 + (q-1)\,]\,N_r'\}$$

$$= \left(M_r' + \frac{n^2-1}{3}\,N_r'\right)$$

(c) The average resistance ratio for the upper coil side of a two-coil-side-per-slot winding having n layers per coil side reduces to

$$K = \left[\;M_r' + \left(\frac{4\,n^2-1}{3} + n^2\cos\theta\right)N_r'\;\right]$$

θ is the phase angle between the currents in the upper and lower coil sides.

If the strands are joined at the beginning and end of a whole turn the loss ratio is a little more difficult to calculate inasmuch as the losses in the two half turns are different. There are two cases to consider, one in which the end connections are not turned over and one in which they are turned over.

Case 2. Strands joined at the beginning and end of a single turn; end connections not turned over. The heating loss in the whole turn may be expressed symbolically as:

$$\text{loss} = 2\,I \,.\, R\,(I\,M' + I_0/2\,N' + I_s\,S'/2) \\ + I_b' \,.\, R\,[\,(I/2 + I_0)\,N' + (I_b' - I_0)\,T' \\ + I/2\,S'] \\ + I_b'' \,.\, R\,[\,(I/2 + I_0)\,N' + (I_b'' - I_0,\,T' \\ + I/2\,S']$$

The first term is the power due to the current in the conductor and the resistance drop in the top element of the top strand of the turn. The second term is the power due to the current, I_b', below the upper half turn and the pressure produced by the flux within this half turn. Similarly the third term is the power due to the current, I_b'', below the lower half turn and the pressure produced by the flux within this half turn. For the q conductor, $I_b' = I\,(q - 1 + n\,\underline{/\,\theta})$ and $I_b'' = I\,(q-1)$. We have also shown that in this case (2) $I_s = I\,(q + n/2\,\underline{/\,\theta})$, and $I_0 = I\,(q - 1 + n/2\,\underline{/\,\theta})$.

As before θ is the phase angle between the currents in the upper and lower coil sides lying in the same slot. Making these substitutions and dividing by the direct-current resistance loss for a whole turn gives a resistance ratio per turn of

$$K = (M_r' + (q^2 - q + n^2/4 + (2\,q-1)\,n/2\cos\theta)\,N_r' \\ + n^2/4\;\;T_r')$$

The average value of this resistance ratio for a whole coil of n turns is

$$K = \left[\;M_r' + \left(\frac{7\,n^2-4}{12} + n^2/2\cos\theta\right)N_r' \\ + n^2/4\;\;T_r'\;\right]$$

Case 3. Strands joined at the beginning and end of a whole turn; end connections turned over. The heating loss for a whole turn is now:

$$\text{loss} = 2\,I \,.\, R\,(I\,M' + I_0/2\;N' + I_s\;S'/2) \\ + I \,.\, R\,(-\,(I/2 + I_0)\,N' + (I_b'' + I_0 + I)\,T' \\ + I\;\;S'/2 \\ + I_b'\,R\,[\,(I/2 + I_0)\,N' + (I_b' - I_0)\,T' \\ + I\,S'/2\,]$$

$$+ I_b'' \,.\, R\,[-(I/2 + I_0)\,N' + (I_b'' + I_0 + I)\,T' + I\,S'/2]$$

The first term is the power due to the current in the turn and the resistance drop in the top strand of the turn in the upper coil side and the bottom strand of the turn in the lower coil side. Due to the turning over of the end connections these two half-turn strands are a part of the same strand. The second term is the power due to the current in the turn and the pressure produced by the flux within the half turn that is in the lower coil side. The third term is the power due to the current below the half turn in the upper coil side and the pressure acting on this current that is produced by the flux within this half turn. The fourth term is similarly the power due to the current below the half turn in the lower coil side and the pressure acting on this current that is produced by the flux within this half turn. In this case:

$I_0 = (-1/2 + n/2\,\underline{/\,\theta})\,I \quad I_b' = (q - 1 + n\,\underline{/\,\theta})\,I$

$I_s = (q - 1/2 + n/2\,\underline{/\,\theta})\,I \quad I_b'' = (q - 1)\,I$

Making these substitutions the resistance ratio for a whole turn reduces to:

$$K = \left\{ M_r' + \frac{n^2 - 1}{4} N_r' + [(2\,q - 1)^2 + n^2 + 2\,(2\,q - 1)\,n \cos\theta]\,T_r'/4 \right\}$$

The average resistance ratio for a whole coil of n turns is:

$$K = \left[M_r' + \frac{n^2 - 1}{4} N_r' + \left(\frac{7\,n^2 - 1}{12} + n^2/2 \cos\theta \right) T_r' \right]$$

The method of calculating the leakage impedance when the stranding is continuous throughout a whole coil is described in considerable detail in the preceding paper and need not be repeated here. With a finite number of strands, however, there are added terms involving S' which appear on account of the insulation between the strands. Fortunately their effect is not difficult to calculate.

The added resistance drop in a coil of n turns is (equation **19**) $2\,n\,R\,I_s/2\,S'$. Values of I_s have already been calculated for the different cases. The added pressure acting in the coil side which lies in the bottom of the slot due to flux within the coil side above it is $n\,R\,(n\,\underline{/-\,\theta})\,I/2\,S'$. The other added terms are those due to the pressure acting in the conductors of the coil produced by flux within the coil itself. There are three cases, *viz.*, 4, 5 and 6.

Case 4. When the end connections are not turned over the resistance drop taken is that in the upper element of the top strand of the conductors of both coil sides. The added drop is then:

$$\sum_1^n 2\,(q - 1)\,R\,I/2\,S' \quad \text{or} \quad \frac{n\,(n - 1)}{2} R\,I\,S'$$

Case 5. When the end connections are turned over on one side only the resistance drop taken is that in the top element of the top strand of the first half turn plus that in the bottom element of the bottom strand of the next half turn plus that in the bottom element of the bottom strand of the next half turn plus that in the top element of the top strand of the next half turn plus, etc. In this case it is readily shown that the added drop is:

$$\sum_1^n (2\,q - 1)\;R\;I/2\;S' \quad \text{or} \quad n^2/2\;R\,I\,S'$$

Case 6. When the end connections are turned over on both sides the resistance drop taken is that in the top elements of the top strands of one coil side and the bottom elements of the bottom strands of the other coil side. The added pressure is thus

$$\sum_1^n (q - 1)\,R\,I/2\,S' + \sum_1^n q\,R\;I/2\;S') \quad \text{or} \quad n^2/2\,R\,I\,S'$$

The sum of these three component added terms is the same in each of the three cases. It is $(n^2 + n^2 \cos\theta)\,R\,I\,S'$.

The average value for a single half turn is $(n/2 + n/2 \cos\theta)\,R\,I\,S'$.

We may now write the expressions for the slot leakage impedance of a symmetrical pair of fractional pitch slots.

Refer to the preceding paper.

Case 4. End connections not turned over.

$$Z = R_c \left[M' + \left(\frac{2\,n^2 - 1}{4} + n^2/2 \cos\theta \right) N' + \frac{4\,n^2 - 1}{12} T' + (n/2 + n/2 \cos\theta)\,S' \right]$$

where R_c is the true resistance of a whole coil.

Case 5. End connections turned over on one side only. Even number of conductors per coil side.

$$Z = R_c \left[M' - N'/4 + \left(\frac{10\,n^2 - 1}{12} + n^2/2 \cos\theta \right) T' + (n/2 + n/2 \cos\theta)\,S' \right]$$

Case 5. Odd number of conductors per coil side.

$$Z = R_c \left[M' + \left(\frac{10\,n^2 - 1}{12} + n^2/2 \cos\theta \right) T' + (n/2 + n/2 \cos\theta)\,S' \right]$$

Case 6. End connections turned over on both sides.

$$Z = R_c \left[M' + \frac{n^2 - 1}{4} N' + \left(\frac{7\,n^2 - 1}{12} \right. \right.$$

$$+ n^2/2 \cos\theta \Big) T' + (n/2 + n/2 \cos\theta)\, S' \Big]$$

Since S' is pure imaginary, the terms involving it do not appear in the expressions for the alternating-current resistance. In the preceding paper $\alpha^2 d^2$ which is now replaced by T' was pure imaginary. T', however, has both real and imaginary parts, and thus adds to the value of the alternating-current resistance.

There follows a numerical calculation of the heat losses in a specified winding. The pitch of the coils is one and the dimensions of the slot and the conductors are:

Width of slot (s) .... = 2.54 cm.
Width of conductor (w) .... = 1.60 "
Length of armature core (l_1) .... = 72.4 "
Length of end turn (l_2) .... = 80.0 "
Depth of strand (d/m) .... = 0.254 "
Thickness of insulation between strands (a) .... = 0.0381 "
Number of strands per conductor (m) .. = 7.0
Number of conductors per coil side (n) .. = 2.0
Frequency (f) .... = 60 cycles
Average temperature of winding ...... = 100 deg.cent

ρ = 2260 c. g. s. ohms at 100 deg. cent.

$$\alpha = 2\pi \sqrt{\frac{2wf}{\rho s}}\;\underline{/\,45^\circ} = 1.15\,\underline{/\,45^\circ}$$

$$\frac{\alpha d}{m} = 0.292\,\underline{/\,45^\circ}$$

$$\frac{\alpha d}{2m} = 0.146\,\underline{/\,45^\circ}$$

$$\frac{\alpha a}{2} = 0.0219\,\underline{/\,45^\circ}$$

$$\tanh\frac{\alpha d}{2m} = 0.146\,\underline{/\,44^\circ\,36.5'}$$ [4]

$$\frac{\dfrac{\alpha d}{m}}{\sinh\dfrac{\alpha d}{m}} = 1.00\,\underline{/-0.82}$$ [4]

$$\frac{T'}{4m^2} = \frac{\alpha d}{2m}\cdot\left(\frac{\alpha a}{2} + \tanh\frac{\alpha d}{2m}\right)$$

$$= 0.0245\,\underline{/\,89^\circ\,38.5'}$$

From equation (**10a**)

$$\sinh\frac{\beta d}{2m} = \sqrt{\frac{\dfrac{\alpha d}{2m}\left(\dfrac{\alpha a}{2} + \tanh\dfrac{\alpha d}{2m}\right)}{\dfrac{\dfrac{\alpha d}{m}}{\sinh\dfrac{\alpha d}{m}} + \dfrac{l_2}{l_1}}}$$

$$\sinh\frac{\beta d}{2m} = \sqrt{\frac{0.0245\,\underline{/\,89^\circ\,38.5'}}{1.0\,\underline{/-0.82^\circ} + 1.105}}$$

$$= \sqrt{0.01164\,\underline{/\,45^\circ\,1.5'}}$$

$$= 0.07628 + j\,0.07634$$

Let $$\frac{\beta d}{2m} = g + jk$$

$$\sinh(g + jk) = \sinh g \cos k + j \cosh g \sin k$$

$$= A + jB = C\,\underline{/\tanh^{-1} B/A}$$

That is $\sinh g \cos k = A$

$\cosh g \sin k = B$

Solving these two equations for g and k gives:

$$\sinh g = \sqrt{\frac{-(1 - C^2) + \sqrt{(1 - C^2)^2 + 4A^2}}{2}}$$

and $$\sin k = \sqrt{\frac{1 + C^2 - \sqrt{(1 + C^2)^2 - 4B^2}}{2}}$$

Substitution in these solutions shows that:

$\sinh g = 0.0766$ $g = 0.0767$
$\sin k = 0.0761$ $k = 0.0762$ radians

Therefore $$\frac{\beta d}{2m} = 0.108\,\underline{/\,44^\circ\,50'}$$

$\tanh\dfrac{\beta d}{2m}$ is readily computed in this case from

$$\tanh\frac{\beta d}{2m} = \frac{\sinh\dfrac{\beta d}{2m}}{\sqrt{1 + \sinh^2\dfrac{\beta d}{2m}}}$$

$$= \frac{0.1079\,\underline{/\,45^\circ\,1.5'}}{\sqrt{1 + 0.01164\,\underline{/\,90^\circ\,3'}}}$$

$$= 0.1079\,\underline{/\,44^\circ\,21.5'}$$

$$\alpha d\;\frac{\dfrac{\alpha a}{2} + \tanh\dfrac{\alpha d}{2m}}{\tanh\dfrac{\beta d}{2m}} = 3.18\,\underline{/\,45^\circ\,19'}$$

The values of $\coth \beta d$ and $2\tanh\dfrac{\beta d}{2}$ can be computed from the formulas:

$\coth \beta d$

$$= \frac{\sinh 2mg \cosh 2mg - j \sin 2mk \cos 2mk}{\sinh^2 2mg \cos^2 2mk + \cosh^2 2mg \sin^2 2mk}$$

$2\tanh\dfrac{\beta d}{2}$

$$= 2\,\frac{\sinh mg \cosh mg + j \sin mk \cos mk}{\cosh^2 mg \cos^2 mk + \sinh^2 mg \sin^2 mk}$$

From which

$$\coth \beta d = 0.8835\,\underline{/-11^\circ\,20'}$$

$$2\tanh\frac{\beta d}{2} = 1.472\,\underline{/\,34^\circ\,7'}$$

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[4] From Kennelly's "Tables of Complex Hyperbolic and Circular Functions."

The complex quantities M', N' and T' may now be calculated

$$M' = 1.335 \,\underline{/33^\circ\,59} = 1.11 + j\,0.746$$

$$N' = 2.22 \,\underline{/79^\circ\,26'} = 0.408 + j\,2.19$$

$$T' = 2.28 \,\underline{/89^\circ\,38.5'} = 0.014 + j\,2.28$$

If the number of strands, m, is increased without limit while the depth of the conductor and the relative amount of insulation between the strands are unchanged we have:

$$\sinh \frac{\beta d}{2\,m} \doteq \frac{\beta\,d}{2\,m} \doteq \frac{\alpha\,d}{2\,m} \sqrt{\frac{a/d + 1}{l_2/l_1 + 1}}$$

since $\tanh \dfrac{\alpha\,d}{2\,m} \doteq \dfrac{\alpha\,d}{2\,m}$

and

$$\frac{\dfrac{\alpha\,d}{m}}{\sinh \dfrac{\alpha\,d}{m}} \doteq 1$$

$$M' \doteq \beta\,d \coth \beta\,d$$

$$N' \doteq \beta\,d\,2 \tanh \frac{\beta\,d}{2}$$

For the winding we are considering,

$$\beta\,d \doteq 0.739\,\alpha\,d \doteq 1.51 \,\underline{/45^\circ}$$

For this value of $\beta\,d$; $M_r' = 1.11$ and $N_r' = 0.413$. In this case, since T' is pure imaginary, $T_r' = 0$.

A further calculation has been made for conductors of the same net cross-section, but consisting of three strands instead of seven. In this case:

$$M' = 1.295 \,\underline{/33^\circ\,9'} \qquad M_r' = 1.084$$

$$N' = 2.04 \,\underline{/80^\circ\,52'} \qquad N_r' = 0.324$$

$$T' = 2.086 \,\underline{/87^\circ\,52'} \qquad T_r' = 0.0776$$

The following is a table of the ratios of alternating to direct-current resistance for this winding for each of the six strand arrangements.

Strand Arrangement	Resistance Ratio		
	$m = \alpha$	$m = 7$	$m = 3$
Case 1	3.17	3.15	2.70
" 2	2.76	2.75	2.46
" 3	1.42	1.47	1.66
" 4	2.66	2.66	2.40
" 5	1.01	1.08	1.41
" 6	1.42	1.47	1.66

Discussion

W. V. Lyon (by letter): A series of curves has been plotted which, though they may not be of particular value themselves, indicate a line of investigation that should prove of considerable importance. Curves similar to them have been discussed by Rogowski and others.

The curves marked "I" are for the upper bar of a laminated bar winding, with two bars per slot.

The curves marked "II" are the average values for the slot. Laminated conductors with laminations joined at the beginning and end of each turn. Two turns per coil, two coil sides per slot.

The curves marked "III" are the average values for the slot. The winding is exactly like the preceding case but the end connections are turned over.

The curves marked "IV" are the average values for the slot. Laminated coil with laminations joined at the beginning and end of the coil. Two turns per coil, two coil sides per slot. End connections turned over.

All of these curves are plotted for full-pitch, finely laminated windings. The length of the armature core is assumed equal to the length of one end connection. The ordinates of these curves show relative values only. The abscissas, $\alpha\,d$, are roughly equal to the depth of the conductor in centimeters for a frequency of 60 cycles.

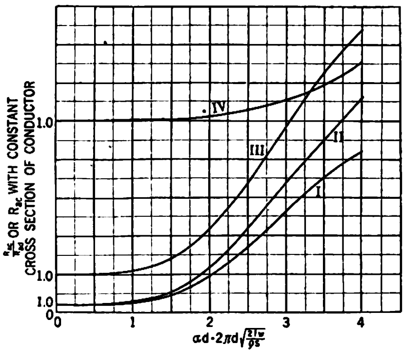

Fig. 1

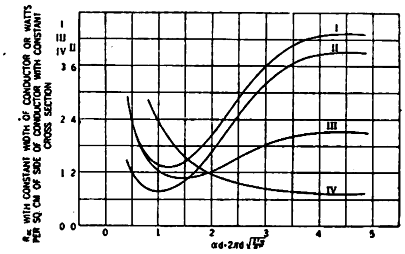

Fig. 2

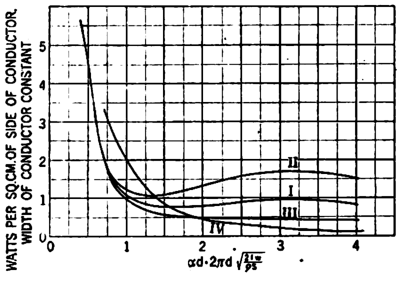

Fig. 3

Fig. 1 shows how entirely inappropriate laminated bar windings are when the conductors are deeper than about one centimeter. Fig. 2 shows that with conductors of constant width there is a depth which makes the alternating-current resistance a minimum. This has been called the critical depth. Notice that with the better types of windings, viz. III and IV, the critical depth is greater. The ordinates of those curves may also represent the watts per square centimeter of coil side for conductors of constant cross section. If all of the heat developed in the conductors passed through the *sides* of the coil, which of course it does not, these curves would show a truly critical depth.

If the width of the conductor is kept constant the watts per square centimeter of coil side do not reach as distinct a minimum value. In fact in the better windings, III and IV there is no minimum. *i. e.*, critical depth. Fig. 3.

Taking the surface through which the heat is conducted from the coil as the entire perimeter of the cross section of the coil the watts per square centimoter of insulation are plotted in Fig. 4. These curves were plotted for cases in which the cross section of the conductors was nine square centimeters and the thickness of the insulation was one-half a centimeter.

It seems that these curves only emphasize the need of a thorough investigation of the heat conduction through the insulation of embedded conductors. The problem is further complicated in that the best shape of conductor must be determined by considering not only the heat generated within it but also by the eddy current and hysteresis losses produced in the neighboring iron.

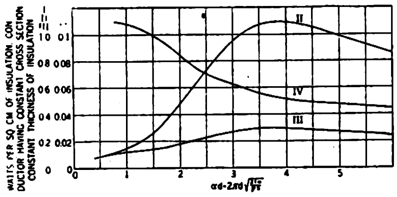

Fig. 4

M. S. Vallarta (by letter): Although a few resistance-ratio and temperature measurements have occasionally been reported, no thorough experimental investigation of the Field theory of skin-effect in embedded conductors has to our knowledge ever been undertaken. In order to fill this gap, such an investigation is now in progress at the Research Division of the Electrical Engineering Department, Massachusetts Institute of Technology. Its main purpose is to furnish experimental proof of the correctness of Field's assumptions, since the rest of the theory and its applications, as developed by Rogowski, Richter, Gilman and very recently by Professor Lyon, in his simple method involving the hyperbolic functions of a complex variable, follow from these assumptions by a process of mathematical deduction. Owing to experimental difficulties it has been found necessary, however, to test both the assumptions and the conclusions, as a check on the complete theory.

A piece of standard laminated slotted armature was kindly furnished for these tests by the Westinghouse Company, to which we acknowledge our indebtedness. This is mounted on a wooden framework, away from any magnetic material. So far as the magnetic circuit is concerned, conditions are in close agreement with the ideal demands of the theory.

For the purpose of obtaining the greatest possible resistance, the test coils are made of thin copper strip wound longitudinally in the slots, with its largest dimension parallel to the slot side and with paper-insulated turns. So far as one-dimensional skin-effect is concerned, which is the only one considered in the Field theory, these coils are the exact equivalent of a solid bar conductor of the same dimensions. Stranded conductors with twisted end connections have not been thus far experimentally investigated.

To test the assumption that an element of current in the slot produces no field below it, an exploring coil was placed longitudinally in the slot, with its plane parallel to the slot side, below a current carrying conductor. This exploring coil was connected through a shielded two-step vacuum tube amplifying circuit and thermocouple to a sensitive galvanometer. No deflection could in any case be obtained, thus confirming the assumption. No satisfactory proof of Field's assumption of no component of field strength parallel to the slot side has been found.

An experimental curve of cross-flux distribution was determined by winding five equi-distant exploring coils around the tooth and measuring the voltage induced in them when an alternating current flows through the slot conductors. This voltage was measured by connecting the exploring coils in turn to a sensitive galvanometer through a shielded two-step vacuum tube amplifying circuit and thermocouple. To eliminate tooth-tip leakage, an exploring coil at the top of the slot was always connected in series opposition with the coil through which the flux was measured.

The curve of cross-flux distribution is easily calculated from equation four of Professor Lyon's first paper. The flux linking an exploring coil distance x from the bottom of the upper conductor is

$$\phi_1 = \int_x^d \frac{4\,\pi\,l}{s}\left[\int_0^x w\,c\,d\,x + I_2\right] d\,x$$

where $c = f\,(x)$ is given by the equation quoted above, *i. e.*,

$$c = \alpha/w\left[I_1\,\frac{\cosh\alpha\,x}{\sinh\alpha\,d} - I_2\tanh\frac{\alpha\,d}{2}\cosh\alpha\,x + I_2\sinh\alpha\,x\right]$$

It follows by integration, if $I_1 = I_2 = I$.

$$\phi_1 = \frac{4\,\pi\,l\,I}{\alpha\,s}\left[\coth\alpha\,d - \frac{\cosh\alpha\,x}{\sinh\alpha\,d} - \tanh\frac{\alpha\,d}{2}(\cosh\alpha\,d - \cosh\alpha\,x) + \sinh\alpha\,d - \sinh\alpha\,x\right].$$

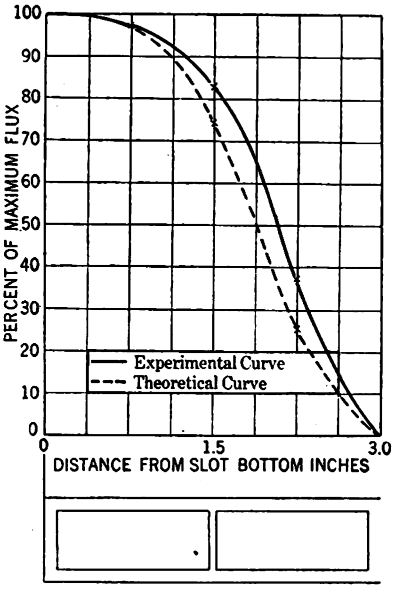

Fig. 5

For the lower conductor alone:

$$\phi_2 = \int_x^d \frac{4\,\pi\,l}{s}\left[\int_0^x w\,c\,d\,x\right] d\,x$$

with

$$c = \frac{\alpha\,I}{w}\;\frac{\cosh\alpha\,x}{\sinh\alpha\,d}$$

whence

$$\phi_2 = \frac{4\,\pi\,l\,I}{\alpha\,s}\left[\coth\alpha\,d - \frac{\cosh\alpha\,x}{\sinh\alpha\,d}\right]$$

Let ϕ_{10} be the total flux linking an exploring coil at the bottom of the upper conductor. The total flux linking an exploring coil distance x from the bottom of the slot is:

$$\phi = \phi_{10} + \phi_2$$

From which the flux at any point along the lower conductor can be calculated.

The experimental and calculated curves of cross-flux distribution are given in Fig. 5. The agreement is considered satisfactory, within the limits of experimental error.

Resistance-ratio measurements have also been made. Power is measured by the three-voltmeter method, especial precautions being taken to have a sinusoidal voltage wave. Oscillographic records show that this condition was in every case closely fulfilled. The disturbing effect of uneven temperature distribution is largely eliminated by operating only after a steady uniform temperature, such as caused by a direct current, has been reached, then making all a-c. measurements within a short interval. To correct for iron loss, a coil of fine magnet wire was wound around the teeth, the excitation being distributed so as to correspond as closely as possible with conditions in the slot conductors. Phase difference effects of course cannot be imitated with this arrangement. The skin effect of such a winding is assumed to be negligible.

Results of resistance-ratio measurements and computation data are given below. Each power measurement given is the average of at least five runs. The precision of individual measurements is estimated at one per cent or better.

COMPUTATION DATA

Dimensions of slot:	Width	1.905	cm.
	Length	40.6	cm.
	Depth	7.62	cm.
Effective width of conductor		1.418	cm.
Effective height of conductor		3.58	cm.
	Frequency	60.0	cycles
	Temperature	65	deg. cent.

TABLE I

Measured power watts	Measurements for two conductors in series			Net power watts	A-C. resistance ohms
	Current amperes	Correction for end turns watts[1]	Correction for iron loss watts		
204.9	9.14	3.84	6.05	195.0	2.34
68.0	5.21	1.22	2.21	64.6	2.38
11.53	2.131	0.20	0.50	10.83	2.39
3.32	1.136	0.05	0.22	3.05	2.38
	Average				2.37 ohms
	Embedded portion d-c. resistance				0.2251 ohms
	Resistance ratio				10.52
	Calculated resistance ratio				10.26
	Difference				2.5%

TABLE II[2]

Measured power watts	Measurements for lower conductor			Net power watts	A-C. resistance ohms
	Current amperes	Correction for end turns watts[1]	Correction for iron loss watts		
95.8	15.16	5.25	3.14	87.4	0.382
34.9	9.16	1.92	1.21	31.7	0.378
11.81	5.32	0.65	0.37	10.79	0.381
1.98	2.21	0.09	0.05	1.84	0.377
	Average				0.3795 ohms
	Embedded portion d-c. resistance				0.1148 ohms
	Resistance ratio				3.30
	Calculated resistance ratio				3.25
	Difference				2.0%

TABLE III[3]

Measured power watts	Measurements for upper conductor			Net power watts	A-C. resistance ohms
	Current amperes	Correction for end turns watts[1]	Correction for iron loss watts		
181.5	9.36	2.15	2.36	177.0	2.04
62.7	5.47	0.74	1.54	60.4	2.01
10.17	2.145	0.11	0.46	9.60	1.98
2.70	1.140	0.03	0.07	2.60	2.00
	Average				2.01 ohms
	Embedded portion d-c. resistance				0.1110 ohms
	Resistance ratio				18.1
	Calculated resistance ratio				17.27
	Difference				4.8%

It is seen from the above that, as far as can be judged from our present evidence, the complex hyperbolic method developed by Prof. Lyon correctly describes the cross-flux distribution and gives simple means of computing the resistance ratio of ideal bar windings with engineering accuracy.

In connection with this investigation, the following bibliography has been prepared:

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19. A. E. Gilman. "Eddy-Current Losses in Armature Conductors." TRANS. A. I. E. E., Vol. XXXIX, 1920, Part I, p. 997.

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[1] Length of end turns is 20 per cent of total length. Correction assumes that skin effect in air for the coils is negligible, which is borne out by experiment.

[2] Lower conductor alone in slot.

[3] Both conductors carrying the same current.

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V. Karapetoff: Prof. Lyon deserves much credit for having worked out in detail various expressions for heat losses in armature conductors, with a non-uniform distribution of current. I wish only to point out some possible improvements in the fundamental mathematical treatment, and to indicate the next steps that might be taken in the solution of this important problem.

1. *Gilman's Work.* In June 1920 Mr. R. E. Gilman presented before the Institute a highly mathematical paper on the same subject (TRANS. Vol 39, p. 997), and it is to be regretted that in Prof. Lyon's paper, presented a year and a half later, before the same body, no attempt is made to correlate his work with that of Gilman's, and to point out identical results, discrepancies, if any, advantages and disadvantages of the two different methods of mathematical analysis. It remains for a future investigator to finish this part of the work.

2. *Exponential Notation.* In my discussion of Gilman's paper (*ibid*, p. 1054) the advantages of exponential notation are pointed out, and it is shown that several pages of long tedious formulas may be done away with, and that a solution of certain simultaneous equations is obviated. It now remains to compare the method of exponential notation with the use of hyperbolic functions of a complex variable, as used in Prof. Lyon's paper. In making this comparison, it is necessary to keep in mind that at the present time we have only Kennelly's tables of such functions, with steps of such magnitude that the two-directional interpolation is quite tedious. The building up of hyperbolic functions of a complex variable out of those of real variables also takes considerable time.

3. *The fundamental differential equation.* The fundamental differential equation of distribution of alternating-current density in a long conductor, subjected to a transverse magnetic flux, is not original either with Mr. Gilman or with Prof. Lyon. See, for example, A. Russell, Alternating Currents, Vol. I, index under "eddy currents." This equation can now be derived much more directly from Heaviside's laws of circuitation which are familiar to the younger generation of engineers. We want as wide a circle of readers for our Institute papers as possible, and any simplification in mathematics, any deduction directly from a general physical law, rather than by a special "follow me" method, is a step in the right direction.

4. *Some mathematical simplifications.* Prof. Lyon's paper being based on a well-known fundamental equation, the value of the contribution lies mainly in the application of the solution to certain specific cases. The solution must be in the simplest possible form for numerical computations. An inspection of his formulas shows the possibility of considerable further simplification. The formulas used in the present paper are based on equation (3) in his first paper (TRANS., 1921, Vol. 40, p. 1361). and it is therefore necessary to go back to that paper in order to indicate an alternative treatment.

The objections to Prof. Lyon's formulas are (a) that they are involved and unsymmetrical, and (b) that the variable x enters in the same formula in two or more places. What seems to be a simpler form of solution both for a general development and for numerical work, is indicated below; the equations numbers **1**, **2** and **3**, refer to Lyon's first paper:

Let the general solution of equation (**2**) be written in the form of

$$\underset{\cdot}{c} = (\alpha/w)\ \underset{\cdot}{D}\ \text{Sinh}\ (\alpha\, y + \beta) \qquad \textbf{(10)}$$

This solution differs from equation (**3**) in the following respects: (a) the expression (α/w) is written out explicitly in order to simplify further transformations; (b) the variable distance, y, is assumed to be measured from the center of the conductor, in order to make the equations more symmetrical. In the original paper the corresponding variable, x, is measured from the lower edge; (c) the variable y enters in the equation only once, while equation (**3**) contains x twice; (d) The two constants of integration are D and β, that is, directly the amplitude and the phase, instead of the components $\underset{\cdot}{A}$ and $\underset{\cdot}{B}$.

To determine $\underset{\cdot}{D}$ and β, we substitute expression (**10**) in equation (**1**), and in the formula for I_1*. After simplification we find:

$$\tanh \beta = [I_1/(I_1 + 2\, I_0)]\quad \text{Coth}\ \alpha\, d/2 \qquad \textbf{(11)}$$

$$\underset{\cdot}{D} = \frac{1}{2}\, I_1/(\text{Sinh}\ \beta\ \text{Sinh}\ \alpha\, d/2) \qquad \textbf{(12)}$$

Equation (**10**), with the auxiliary expressions (**11**) and (**12**) leads to simpler formulas and computations than the formulas used in both of Prof. Lyon's papers. For the current density at the center of the conductor we have, putting $y = 0$,

$$\underset{\cdot}{c} = (\alpha/w)\ \underset{\cdot}{D}\ \text{Sinh}\ \beta \qquad \textbf{(13)}$$

or, substituting for $\underset{\cdot}{D}$ its value from eq. (**12**),

$$\underset{\cdot}{c} = \alpha\, I_1/(2\, w\ \text{Sinh}\ \alpha\, d/2) \qquad \textbf{(14)}$$

which checks with Prof. Lyon's expression.†

5. *Future Work.* There is no reason why this problem should not be brought now to a final solution, at least in application to standard turbo-alternators. By this I mean a set of curves, charts, tables, etc. with which a designer could safely compute the armature copper loss for an assumed arrangement of conductors. At the present stage the designer would have to study one or two highly mathematical articles and then perform rather long computations before he could get a specific result. Should he then decide to change his winding, much of the numerical work would have to be repeated.

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*TRANS. A. I. E. E., 1921, p. 1369.

†TRANS. A. I. E. E., 1921, p. 1372.

LIST OF RESEARCH BULLETINS—Continued

Bulletin Number

17. Electromagnetic Theory of the Telephone Receiver, with Special Reference to Motional Impedance. A. E. Kennelly and H. Nukiyama; March, 1919.
18. A Rectangular-Component Two-Dimensional Alternating-Current Potentiometer. A. E. Kennelly and Edy Velander; July, 1919.
19. Current Distribution in Armature Conductors. Waldo V. Lyon; July, 1919.
20. Alternating-Current Planevector Potentiometer Measurements at Telephonic Frequencies. A. E. Kennelly and Edy Velander; July, 1919.
21. A Theory of Metallic Arc Welding. Ralph G. Hudson; March, 1920.
22. Oscillographs and Their Tests. A. E. Kennelly, R. N. Hunter and A. A. Prior; April, 1920.
23. A Direct Recording Method of Measuring Magnetic Flux Distribution. F. S. Dellenbaugh, Jr.; October, 1920.
24. The Transient Process of Establishing a Steadily Alternating Current on a Long Line, from Laboratory Measurements on an Artificial Line. A. E. Kennelly and U. Nabeshima; January, 1921.
25. A Dynamometrical Comparator. A Differential Dynamometer for Accurate Comparison of Alternating and Direct-Current Strengths. Edy Velander, February, 1921.
26. Induction Motor Core Losses. P. L. Alger and R. Eksergian, February, 1921.
27. Acoustic Impedance and Its Measurement. A. E. Kennelly and K. Kurokawa, February, 1921.
28. A Simple Harmonic Analyser. V. Bush; Electromechanical Device for Rapid Schedule Harmonic Analysis of Complex Waves. Frederick S. Dellenbaugh, Jr.; Alignment Chart for Circular and Hyperbolic Functions of a Complex Argument in Rectangular Coördinates. V. Bush, January, 1922.
29. A Mechanical Frequency-Meter of Telephonic Range. A. E. Kennelly and C. H. Manneback, January, 1922.
30. An Integral Equation for Skin Effect in Parallel Conductors. Charles Manneback, May, 1922.
31. The Effect of Absorbed Gas on the Conductivity of Glass. V. Bush and L. H. Connell, September, 1922.
32. On the Education of Engineers in France. A. E. Kennelly, January, 1923.
33. Heat Losses in Conductors. W. V. Lyon, March, 1923.

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F. S. Dellenbaugh, Jr.

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LIST OF RESEARCH BULLETINS

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*1. The Economical Transportation of Merchandise in Metropolitan Districts. H. Pender and H. F. Thomson; March, 1912.

*2. Notes on the Cost of Motor Trucking. H. Pender and H. F. Thomson; October, 1912.

*3. Observations on Horse and Motor Trucking. H. Pender and H. F. Thomson; March, 1913.

4. Relative Fields of Horse, Electric and Gasoline Trucks. H. F. Thomson; August, 1914.

*5. The Delivery and Handling of Miscellaneous Freight at the Boston Freight Terminals. H. Pender, H. F. Thomson, and C. P. Eldred; February, 1914.

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8. The Mechanics of Telephone-Receiver Diaphragms as Derived from their Motional-Impedance Circles. A. E. Kennelly and H. A. Affel; November, 1915.

9. Experimental Researches on Skin Effect in Conductors, A. E. Kennelly, F. A. Laws and P. H. Pierce; September, 1915.

10. Tractive Resistances to a Motor Delivery Wagon on Different Roads and at Different Speeds. A. E. Kennelly and O. R. Schurig; June, 1916.

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14. Street Railway Fares; their relation to length of haul and cost of service. D. C. Jackson and D. J. McGrath; August, 1917.

15. Apparent Dielectric Strength of Varnished Cambric. A. E. Kennelly and R. J. Wiseman; January, 1918.

16. Magnetic Flux Distribution in Annular Steel Laminæ. A. E. Kennelly and P. L. Alger; March, 1918.

*Out of print.

Another Harmonic Analyzer

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Massachusetts Institute of Technology.

Review of the Subject.*—Modern alternating-current engineering involves the necessity of frequently analyzing curves for harmonics. All of the methods of analysis have more or less serious objections. A method based upon the averaging of ordinates selected at the proper intervals has been developed into a machine which is reasonably simple and cheap to construct and can be operated without great skill. The machine depends chiefly for its operation upon a bar containing notches at the proper intervals and each bar is suitable for curves of the same plotted wave length. It is thus limited to curves plotted with the same scale of abscissas, but different bars may be manufactured for different scales and the machine thus adopted to various types of curves. A comparison with the times required for harmonic analysis by different methods shows that it compares favorably with other devices.*

CONTENTS

Introduction (430 w.)
Construction of Analyzer (540 w.)
Operation of Analyzer (800 w.)

WHILE working in the Research Division laboratory at Massachusetts Institute of Technology, Mr. David O. Woodbury devised an extremely simple and useful harmonic analyzer, which is thought worth describing as it can be made to analyze harmonics of very high order without the usual attendant exhaustion suffered by the investigator. It is also rapid for harmonics of any order, simple to build, and while it is limited to curves having the same length of base for one period, it can be fitted to any size of curve by merely making a new bar, which must be accurately machined, but is not complicated or expensive.

The theory of harmonic analysis has been gone into very thoroughly in many text books and will not be here repeated. For bibliography the reader is referred to the A. I. E. E. JOURNAL, Vol. XL, p. 135, 1921. The analyzer being described utilizes the method of selected ordinates, originally proposed by Houston-Kennelly, and developed in detail by Fischer-Hinnen. The curve must be divided into a different number of portions for each harmonic, the number of parts being the same as the order of the harmonic, In this way the sine or cosine multipliers of the usual schedule form become unity (or zero) and the value of the coefficient for the particular harmonic may be obtained by simply averaging ordinates at the different points given by the division of the curve into parts. This may be expressed as follows:

FOR ODD HARMONICS ONLY (180 DEG. OF CURVE REQUIRED)

If, starting at $X = 0$, we measure n ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes of the nth, 3nth, 5nth............*Cosine components.*

If, starting at $X = \pi/2n$, we measure n ordinates at intervals of π/n the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes, taken alternately plus and minus, of the nth, 3nth, 5nth........*Sine components.*

FOR BOTH ODD AND EVEN HARMONICS. (360 DEG. OF CURVE REQUIRED)

If, starting at $X = 0$, we measure $2n$ ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes of the nth, 3nth, 5nth,..........*Cosine components.*

If, starting at $X = \pi/2n$, we measure $2n$ ordinates at intervals of π/n, the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes, taken alternately plus and minus, of the nth, 3nth, 5nth........*Sine components.*

CONSTRUCTION OF ANALYZER

The analyzer consists in principle of a bar which is free to move in the direction of ordinates but fixed in the direction of abscissas. This bar contains a series of notches spaced 1/3rd, 1/5th or any interval of π/n for a distance equal to the length of curve necessary to explore for analysis. By rotating the bar any series of notches may be brought uppermost. An index carrying cross lines slides upon this bar and bears a spring-operated dog, which drops into the notches sufficiently to locate the position of the index, but lightly enough so that it may be pushed along without effort. The carriage upon which the bar is mounted has a magnetic clutch which engages a scale, the motion being imparted to the scale by the magnetic clutch, and a brake preventing motion of the scale when the circuit of the magnetic clutch is open. If the curve to be analyzed is now placed upon a platen under the index the procedure given in the rules quoted above may be followed mechanically by merely moving the index from notch to notch, bringing the cross lines into alinement with the curve for each position, and properly operating the magnetic clutch so that the respective ordinates are alternately added and subtracted upon the scale. The scale reading will then give the sum of the nth, 3nth, 5nth, etc. harmonic coefficients.

The mechanism may obviously be made either to have a sliding motion in a plane or rotate around centers. A sketch of an early model operating upon the first principle is shown in Fig. 1. The scale is a steel rule laid across the top. The magnetic clutch is merely an old bell magnet running on two round rods. This particular machine was arranged to take a cardboard template of the curve, the motion being imparted to the clutch by a small roller following the template. The clutch was operated by a contact roller traveling over a specially prepared plate with alternate spaces of insulation and metallic conductor, which took the place of the notched bar. Different plates were used for each order of harmonic. This machine was intended to be practically automatic so that the roller was dragged over the curve template slowly and the coefficient read off the scale after each traverse. Mechanical difficulties made this type impractical, so an improved model was made, the parts swinging around centers. This is shown in Figs. 2 and 3.

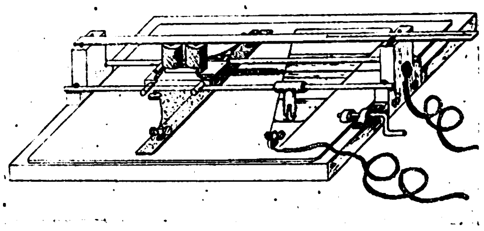

Fig. 1—Early Model of Woodbury Analyzer

The template under roller at right hand side of base is changed for each harmonic.

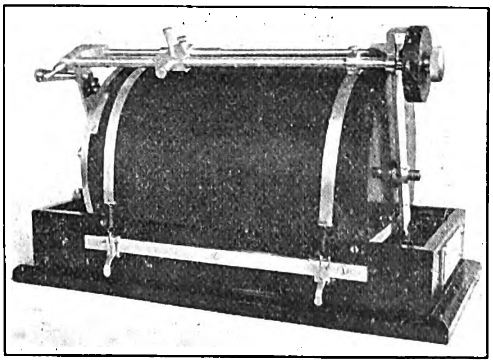

Fig. 2—Front View of Final Model of Woodbury Analyzer

The scale upon which the coefficients are read may be seen abutting the left hand end of cylindrical platen.

The curve is clamped upon the cylindrical platen by means of the two bands. The notching bar may be seen with the setting head upon the right hand end, Fig. 2. The index with cross lines does not show plainly but is carried upon the forward part of the carriage riding upon the notching bar and plain guide bar. The magnetic clutch is attached to the left hand radius supporting arm, showing in right hand end of Fig. 3, and the scale is adjacent to the left hand end of the cylindrical platen almost in contact with clutch. The index for the scale is attached to the left hand side of platen, Fig. 2, and the brake is within the box and may be adjusted by the thumb screw shown in Fig. 3 under right hand strap. The clutch is operated by a switch in the handle upon the left hand radial supporting arm, Fig. 2.

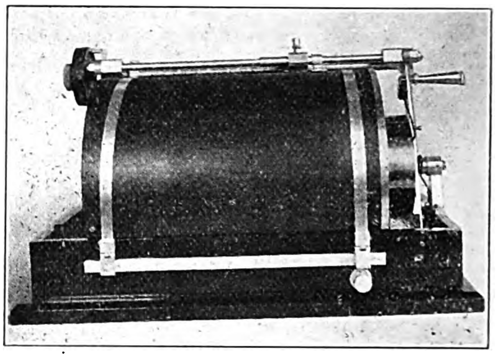

Fig. 3—Rear View of Final Model of Woodbury Analyzer

The small cylindrical drum at the right is the magnetic clutch. The thumb screw under right hand strap is the brake adjustment to prevent slipping of scale.

Operation of Analyzer

To perform the analysis, a notching bar matching the curve to be analyzed is placed in the machine. If the curves are many different scales this, of course, is a serious objection as they must be reduced to the same scale, but if oscillograms are made with a synchronous film drive, or if the curves are plotted by hand from tabular data, the scale can be always made the same and therefore one bar is sufficient unless very many harmonics are desired. The development of the notching for a bar to be used in analyzing a curve with a scale of 6 in. = 180 deg. covering odd harmonics from the 3rd to the 21st is shown in Fig. 4, and the bar may be constructed out of half inch stock.

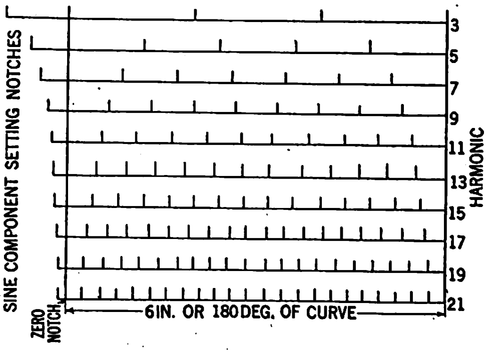

Fig. 4—Developments of Notching Bar for Odd Harmonics Up to 21st

The curve is next adjusted so that its axis is parallel to the notching bar, readily determined by sliding the index with the bar fixed, and the index adjusted so that when the dog is in the zero notch, the cross lines correspond with the zero point on the curve. The index is mounted upon a sleeve sliding within the carriage for this purpose, which is fastened by a set screw when in the right place. The setting head is then turned to the harmonic desired and the carriage moved until the dog falls in the first notch to right of zero. The index is then registered with the curve, the magnetic clutch switch closed, the index then moved until the dog falls into the next notch, registered with the curve, and the clutch switch released. The carriage is then moved to the next notch and the same sequence repeated. With a small amount of practise this manual sequence of events becomes practically automatic and the analysis will proceed as swiftly as it is possible to slide the carriage from notch to notch. The reading upon the scale will then be the sum of the amplitudes of the nth, 3nth, 5nth, etc. *Cosine terms.*

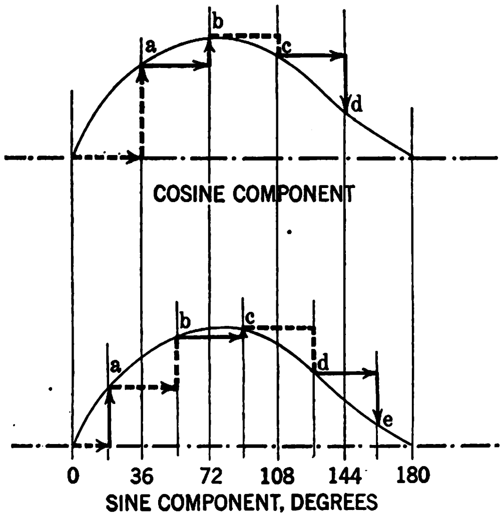

Fig. 5—The Heavy Line Shows the Travel of Analyzer Index.

During the solid portions the clutch switch is closed and during the dotted portions the clutch is released. Any path may be followed by the index between the points of the curve A. B. C. etc. These points are located by the notching bar.

The scale of the result depends upon the divisions made upon machine and reads direct in inches or centimeters. For the *sine* terms the carriage is moved to the special setting notch shown in Fig. 4, to the left of the zero notch, the index set screw is loosened, the index set to the zero of the curve, the carriage moved to the zero notch, and then a similar procedure to that for the cosine term followed. The motion of the index and the closing of the clutch switch for the 5th harmonic is indicated diagrammatically in Fig. 5. The results can be figured in per cent of fundamental, or if scale of curve is known, they can be readily transferred to terms of the function represented.

The fundamental is obtained from the values of the harmonics and from a few specially read ordinates. The following equations show how this is done:

Both Odd and Even Harmonics

$$Y_0 = b_0 + b_1 + b_2 + b_3 \ldots\ldots$$

$$\text{Hence} \quad b_0 = Y_0 - (b_1 + b_2 + b_3 \ldots\ldots)$$

$$1/2\,(Y_0 - Y_{180}) = b_1 + b_3 + b_5 \ldots\ldots$$

$$\text{Hence} \quad b_1 = 1/2\,(Y_0 - Y_{180}) - (b_3 + b_5 \ldots\ldots)$$

$$1/2\,(Y_{90} - Y_{270}) = a_1 - a_3 + a_5 - a_7 \ldots\ldots$$

$$\text{Hence} \quad a_1 = 1/2\,(Y_{90} - Y_{270}) - (-\,a_3 + a_5 - a_7 \ldots)$$

Odd Harmonics Only

$$b_0 = 0$$

$$Y_0 = b_1 + b_3 + b_5 + b_7 \ldots\ldots = 0$$

$$\text{Hence} \quad b_1 = -\,(b_3 + b_5 + b_7 \ldots\ldots)$$

$$Y_{90} = a_1 - a_3 + a_5 - a_7 \ldots\ldots$$

$$\text{Hence} \quad a_1 = Y_{90} - (-\,a_3 + a_5 - a_7 \ldots\ldots)$$

In order to interpret the results it is convenient to have a chart or table in which the readings may be inserted, as otherwise the proper signs, additions and subtractions may become confused. Charts of this sort are shown in Tables I, II and III for both odd and even harmonics.

Conclusions

While this analyzer requires intelligent operation and is not fully automatic, it offers the advantages of simplicity, ease of construction, and rapidity of operation. It also frees the operator from the extremely wearisome amount of multiplication, addition and subtraction required by any extensive analysis with the usual schedule methods, and any one reading may be checked in a few moments without going through a series of calculations. With similar curves, as for instance the magnetizing current of a transformer at different densities, where the general character of the analysis is well known, but the actual magnitudes of the harmonics are desired, very great speed may be obtained, about five minutes being required for the determination of the third, fifth, seventh and ninth harmonics. Table I shows some comparative times required. The accuracy is about the same as that obtained by schedule analysis unless a large number of ordinates is used. In a number of test analyses of curves of known composition, or by comparison with other methods of analysis an average maximum error of about 2 per cent was found, the error being expressed in per cent of the amplitude of the fundamental. With some types of curves slightly larger errors may occur. If the notching bar is carefully constructed the only chance for mechanical errors lies in the slipping of the clutch or dragging of the scale, and as the adjustment of these parts is simple and not at all critical this can not readily occur. Recently in the Research Division Laboratories analysis of all odd harmonics up to and including the 41st were made quite easily, the time required not being measured, but being far less than that necessary for any computational method, and the personal wear and tear being negligible instead of extreme. In general it might be stated that for certain classes of work this analyzer is extremely convenient and can be easily manufactured. It still is not the ideal analyzer but must be used with an understanding of its limitations, under which conditions it is very satisfactory.

TABLE I
TABULATION OF ANALYZER DATA
For Odd and Even Harmonics and Constant Term

Cosine Coefficients

Order of Harmonic	Analyzer Reading	Divide Reading by	Giving	Then Subtract	Net Harmonics		Sum for b_1
					+	−	
b_2		4		$b_6 + b_{10}$			—
b_3		6		b_9			$+ b_3 =$
b_4		8		b_{12}			—
b_5		10		*			$+ b_5 =$
b_6		12		*			
b_7		14		*			$+ b_7 =$
b_8		16		*			—
b_9		18		*			$+ b_9 =$
b_{10}		20		*			—
b_{11}		22		*			$+ b_{11} =$
b_{12}		24		*			—

Sum (B) = ...

*If known harmonics present greater than 12th subtract $b_{3nth} + b_{5nth}$ etc.

Sine Coefficients

Order of Harmonic	Analyzer Reading	Divide Reading by	Giving	Then Subtract	Net Harmonics		Sum for a_1
					+	−	
a_2		4		$-a_6 + a_{10}$			—
a_3		6		$- a_9$			$- a_3 =$
a_4		8		$- a_{12}$			—
a_5		10		*			$+ a_5 =$
a_6		12		*			—
a_7		14		*			$- a_7 =$
a_8		16		*			—
a_9		18		*			$+ a_9$
a_{10}		20		*			—
a_{11}		22		*			$-a_{11}$
a_{12}		24		*			—

Sum (A) = ...

*If known harmonics present greater than 12th subtract $- a_{3nth} + a_{5nth}$ etc.

Y_0 = ... $b_0 = Y_0 - \text{Sum } B$

Y_{90} = ... $b_1 = 1/2\ (Y_0 - Y_{180}) - \text{Sum } B$

Y_{180} = ... $a_1 = 1/2\ (Y_{90} - Y_{270}) - \text{Sum } A$

Y_{270} = ...

Note: Y_0. Y_{90}, Y_{180}, etc. means the values of ordinates at 0 deg., 90 deg., 180 deg., etc.

Equation of curve then:—

$$Y = b_0 + a_1 \sin \theta + a_2 \sin 2\theta + a_3 \sin 3\theta + a_4 \sin 4\theta \ldots\ldots$$
$$+ b_1 \cos \theta + b_2 \cos 2\theta + b_3 \cos 3\theta + b_4 \cos 4\theta \ldots\ldots$$

TABLE II
TABULATION OF ANALYZER DATA
For Odd Harmonics Only

Cosine Coefficients

Order of Harmonic	Analyzer Reading	Divide Reading by	Giving	Then Subtract	Net Harmonics	
					+	−
b_3		3		$b_9 + b_{15}$		
b_5		5		b_{15}		
b_7		7		*		
b_9		9		*		
b_{11}		11		*		
b_{13}		13		*		
b_{15}		15		*		
b_{17}		17		*		

Sum (B) = ...

*If known harmonics present greater than 12th subtract $b_{3nth} + b_{5nth}$. etc.

TABLE II—(*Continued*)

Sine Coefficients

Order of Harmonic	Analyzer Reading	Divide Reading by	Giving	Then Subtract	Net Harmonics		Sum for a_1
					+	−	
a_3		3		$-a_9 + a_{15}$			$- a_3 =$
a_5		5		$-a_{15}$			$+ a_5 =$
a_7		7		*			$- a_7 =$
a_9		9		*			$+ a_9 =$
a_{11}		11		*			$- a_{11} =$
a_{13}		13		*			$+ a_{13} =$
a_{15}		15		*			$- a_{15} =$
a_{17}		17		*			$+ a_{17} =$

Sum (A) = ...

*If known harmonics present greater than 12th subtract $b_{3nth} + b_{5nth}$ etc.

Y_{90} = ... $b_1 = - \text{Sum } B$

$a_1 = Y_{90} - \text{Sum } A$

Note Y_{90} means value of ord. at 90 deg.

Equation of curve:—
$$Y = a_1 \sin \theta + a_3 \sin 3\theta + a_5 \sin 5\theta \ldots\ldots$$
$$+ b_1 \cos \theta + b_1 \cos 3\theta + b_5 \cos 5\theta \ldots\ldots$$

TABLE III
COMPARISON OF TIME REQUIRED FOR HARMONIC ANALYSIS BY DIFFERENT METHODS

Method	Time	No. Harmonics solved	Minutes per Coeff.	Authority
Steinmetz..........	10 hr.	10	60.00	D. C. Miller
Schedule...........	3 hr.	8	22.5	D. C. Miller
Schedule...........	1 hr.	8	7.5	F. W. Grover
Schedule...........	2.5 hr.	17	10.6	Dellenbaugh
Schedule...........	15 min.	3	5.0	D. C. Miller
Coradi Mch........	13 min.	10	1.3	D. C. Miller
Coradi Mch........	7 min.	5	1.4	D. C. Miller
Schedule...........	30 min.	6	5.0	Dellenbaugh
Electric Mch.......	3.5 min	6	0.6	Dellenbaugh
Woodbury Mch.	5 min.	3	1.7	Dellenbaugh
Woodbury Mch.....	4.6 min.	3	1.5	Woodbury
Woodbury Mch.....	9.75min.	6	1.63	Woodbury
Schedule...........	7 min	3	2.3	Woodbury
Schedule...........	22 min.	6	3.7	Woodbury

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20. Alternating-Current Planevector Potentiometer Measurements at Telephonic Frequencies. A. E. Kennelly and Edy Velander; July, 1919.
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24. The Transient Process of Establishing a Steadily Alternating Current on a Long Line, from Laboratory Measurements on an Artificial Line. A. E. Kennelly and U. Nabeshima; January, 1921.
25. A Dynamometrical Comparator. A Differential Dynamometer for Accurate Comparison of Alternating and Direct-Current Strengths. Edy Velander, February, 1921.
26. Induction Motor Core Losses. P. L. Alger and R. Eksergian, February, 1921.
27. Acoustic Impedance and Its Measurement. A. E. Kennelly and K. Kurokawa, February, 1921.
28. A Simple Harmonic Analyser. V. Bush; Electromechanical Device for Rapid Schedule Harmonic Analysis of Complex Waves. Frederick S. Dellenbaugh, Jr.; Alignment Chart for Circular and Hyperbolic Functions of a Complex Argument in Rectangular Coördinates. V. Bush, January, 1922.
29. A Mechanical Frequency-Meter of Telephonic Range. A. E. Kennelly and C. H. Manneback, January, 1922.
30. An Integral Equation for Skin Effect in Parallel Conductors. Charles Manneback, May, 1922.
31. The Effect of Absorbed Gas on the Conductivity of Glass. V. Bush and L. H. Connell, September, 1922.
32. On the Education of Engineers in France. A. E. Kennelly, January, 1923.
33. Heat Losses in Conductors. W. V. Lyon, March, 1923.
34. Another Harmonic Analyser. F. S. Dellenbaugh, Jr., March, 1923.

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*3. Observations on Horse and Motor Trucking. H. Pender and H. F. Thomson; March, 1913.

4. Relative Fields of Horse, Electric and Gasoline Trucks. H. F. Thomson; August, 1914.

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11. Some Properties of Vibrating Telephone Diaphragms. A. E. Kennelly and H. O. Taylor; April, 1916.

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16. Magnetic Flux Distribution in Annular Steel Laminæ. A. E. Kennelly and P. L. Alger; March, 1918.

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Radiation from Transmission Lines*

·BY CHARLES MANNEBACK
Electrical Engineer, Brussels, Belgium

Review of the Subject:— *The aim of this investigation was to contribute something to our knowledge of traveling electromagnetic waves. Among the transient phenomena that occur along transmission lines, these are still little known, especially as far as their "attenuation" and "distortion," or change in shape near the "wave front" are concerned.*

It is first seen that the "classical theory" of the propagation of electrical disturbances along lines, as it has been chiefly developed by Heaviside and Poincare, does not give a correct representation of the facts near the wave front, because it assumes an instantaneous penetration of the current in the wires. It is shown that this theory is an "unidimensional" one, as it considers only one space variable, the coordinate along the line, and, from an electromagnetic point of view, amounts to identifying the traveling waves with plane wave phenomena.

Steinmetz's theory of the radiation from traveling waves (Trans. *A. I. E. E. Feb., 1919) is then examined, and, as Carson pointed out* (Jour. *A. I. E. E., Oct. 1921), found based on a misconception of the propagation of the electromagnetic field near the wires of a line. It is remarked that this theory amounts to propagating longitudinal electric waves, a conception in conflict with the basis of Maxwell's theory. This latter proves very easily that, along a perfect line, i. e., without ohmic or leakage losses, plane electromagnetic traveling waves are propagated without distortion and without attenuation; hence that there is no radiation. Losses do not change anything from the radiation point of view (Mie. Ann. Phys. 2, 1900). In the theory of Steinmetz, the radiation was the controlling factor at high frequencies.*

Although the question could be considered as settled, in so far as Steinmetz raised it, it is felt that the conditions under which traveling waves are started must be elucidated, in order to decide whether, during the transient process of establishing a plane electromagnetic traveling wave, any loss of energy can occur by radiation in free space. This would reduce the radiation to a transient phenomenon instead of to a steady one, as assumed by Steinmetz, i. e., to an effect produced when the condition of plane wave is departed from, as at the origin or the end of a line ("end effect").

The problem, considered in its broad aspect, i. e., from the electromagnetic point of view of Maxwell, appears to be of great complexity. It involves a study of what might be called time-three-dimensional space transient, while the most complicated transients considered in electrical engineering are time-one-dimensional space transients, as was already noticed. It was found possible, however, to decide whether the radiation is a factor of enginering importance in attenuating and distorting waves.

There, first, the distribution of current along the conductors of a line is shown to be a very close approximation to the actual, unknown, distribution, with regard to the possibly existing radiation. Then, from this assumption, the radiation of the system is calculated at a corresponding approximation, and its amount found to be negligible compared to the heat dissipated in the line during the same time. From this it is concluded that the effect of the radiation upon the attenuation and the distortion of the waves along the line must also be negligible, compared to the effect of the joulian losses, and in this result a proof "a posteriori" of the correctness of the initial assumption is seen. The first idea of that procedure, but limited to stationary waves, is probably to be credited to M. Abraham (Phys. Zeit. 2-p. 329-1901) who applied it to the calculation of the radiation from a single isolated wire (oscillator), and found it in practical agreement with a more elaborate theory, based on Maxwell's equations. As in this latter application the radiation is larger than the ohmic dissipation, the conclusions of this investigation are even strengthened.

An application to the steady radiation from a transmission line oscillating freely at one of its natural frequencies shows, even at very high frequencies, that the power radiated is negligible compared to the heat dissipated. For instance, a 100-kilometer transmission line, No. 00 B. & S. wire, oscillating at three million cycles per second, wastes by radiation only 1/8600 of what is wasted by heat. This is in accordance with the results given for steady a-c. traveling waves along a line by Carson, (loc. cit.).

In the case of a traveling wave suddenly started at the origin of a perfect line, it is found that, when the wave has become plane, the amount of energy

$$\frac{3}{2}\, d\, 10^{-9}\, I^2 \text{ joules}$$

is wasted by radiation. d is the distance in cm. between the two wires of the line and I the constant current suddenly flowing in the line, in amperes. This energy is carried to infinity by an electromagnetic field limited to a thin spherical shell of variable depth, (not thicker than d), expanding at the speed of light. A reflection of the wave at a free end of a line is shown to add to the preceding an amount of radiated energy

$$4\, d\, 10^{-9}\, I^2 \text{ joules}$$

and only half that amount for a grounded line. A complete transposition causes a radiation four times as large as the sudden starting of the wave. Compared to the joulian dissipation during the time the wave takes to travel once along the line, the radiated energy is of the order of 1/10,000th.

The influence of the radiation on the attenuation and distortion of traveling waves is thus found entirely negligible in engineering practise. Other possible factors are suggested.

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INTRODUCTION

ONE of the chief reasons at present limiting the development and interconnection of high-power and long-distance electrical transmission lines is the trouble due to the occurrence of abnormal voltages and abnormal currents in the line and the connected apparatus. Whenever any change occurs in

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*This paper is the result of an investigation submitted in partial fulfillment of the requirements for the degree of Doctor of Science from the Massachusetts Institute of Technology, 1922. It has been undertaken under the direction of Professor V. Bush, to whom the author wishes to express his fullest appreciation.

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the steady state conditions of an electrical system, *e. g.* by connecting or disconnecting a load, or by producing or interrupting a short circuit, the new steady state conditions are generally reached after a very short time, during which so-called "transient phenomena" or "transients" take place. As the flow of effective and reactive power are not the same in the different parts of the system during the first and the second steady state, the necessary adjustment is made by transient electromagnetic "traveling waves," *i. e.* waves of both voltage and current, carrying energy from one part of the system to another.

The values of voltage and current in these waves, or as we shall say the amplitude of the waves, may reach abnormally high values and become destructive to the line. The steepness of the front of the wave also, *i. e.*, the high rate of change of the voltage and current with the time may be harmful. Therefore, a knowledge of the "attenuation" or diminution of amplitude of the wave with the time, and of the "distortion" or change in the wave shape, is of engineering importance. Our present knowledge of the subject is still far from being complete. Joulian and dielectric losses produce both attenuation and distortion, but there may be other causes. C. P. Steinmetz[1] considered the possible influence of radiation, but his results may be questioned.

From this began the present investigation, aiming towards the determination of the radiation effect upon the decay of traveling waves along transmission lines. It was found that the radiation effect is entirely negligible compared with the effect of joulian dissipation, and hence that, from an engineering point of view radiation can be ignored in connection with transmission lines.

CLASSICAL THEORY OF THE PROPAGATION OF WAVES ALONG ELECTRIC LINES

The classical theory of the propagation of waves along a line, *i. e.*, a system of straight parallel outgoing and return conductors, rests upon two assumptions: the displacement currents in the dielectric, parallel to the conductors are negligible, and the so-called "four distributed constants" of the line can be determined. They are the resistance, inductance, leakance and capacity per unit length of the line (R, L, G, C). These quantities are functions of the frequency for sinusoidal steady alternating current; for transient currents, they are somewhat indeterminate, and an average value must be assumed.

The fundamental equation for the propagation of the voltage V or the current I is under these assumptions

$$L C \frac{\partial^2 V}{\partial t^2} + (R C + L G) \frac{\partial V}{\partial t} + R G V = \frac{\partial^2 V}{\partial x^2} \tag{1}$$

where t denotes the time and x the space coordinate along the line. From this equation, well known conclusions can be drawn, of which the following only need be considered here:

1. An electric disturbance, *i. e.*, a discontinuity of voltage or current, is always propagated along any line at the constant speed $v = 1 : \sqrt{L C}$, whether there is resistance and leakance or not.
2. The disturbance is attenuated exponentially with the time according to
$$\exp\left[-1/2\,(R/L + G/C)\,t\right].$$
3. The distortion of the disturbance depends on the value of the term
$$1/2\,(R/L - G/C)\ .$$

In the case of transmission lines, the attenuation is the controlling factor, and the wave has died out before any appreciable distortion has taken place. Thus a steep wave front will very nearly keep its shape during the propagation.

This conclusion, however, is based on the two assumptions made; the first amounts to saying that the variations of V and I along the line will not be too rapid, and the second, that the penetration of the current in the wires is instantaneous, neither of which is true at the very front of the wave. From Maxwell's point of view, the classical theory amounts to a plane wave theory, *i. e.*, the electromagnetic field around the wires lies in planes; this implies that the dissipation is taking place everywhere in the plane wave, instead of being localized in the wires. This has been shown very clearly by O. Heaviside in his "Electrical Papers."

We thus consider the question of change in the wave shape near the front of the wave, due to the resistance of the line, still as an open question.

DR. STEINMETZ'S THEORY OF THE RADIATION

Dr. Steinmetz's theory of the radiation from transmission lines deals only with the electromagnetic field. The electric and the magnetic fields are supposed to propagate from the wires perpendicularly into space, thus implying a flow of energy from the wires into space, and also non-transversal electromagnetic waves, two conceptions in conflict with Maxwell's theory. It is shown[2], as an immediate deduction of Maxwell's equations that plane electromagnetic transversal waves are propagated undistorted and unattenuated, at the speed of light in the surrounding medium, along a pair of infinitely long straight perfect conductors, *i. e.*, along a line without joulian or leakage losses; thus there is no radiation. Non-perfect conductors, as it has been shown, introduce more complexity, because the waves are no more plane waves, but nothing is changed from the radiation point of view.[3] The flow of energy in the dielectric is evidently parallel to the wires, with a small component directed into the wires if there are losses. This settles entirely the question, in so far as Dr. Steinmetz raised it: *traveling plane* waves along a line are *not* attenuated nor distorted by radiation, *i. e.*, by escape of energy in a direction perpendicular to the line. Plane waves were an *implicit* assumption in the criticized theory; however, very near the source of a disturbance, or near the end of the line where reflection occurs, or in general near any place where there is a change in the geometrical or electrical behavior of the line, the electromagnetic field, of course, cannot be plane.

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[1] C. P. Steinmetz—PROC. A. I. E. E., Feb. 1919, p. 249.
—Transient Electric Phenomena, 1920; Sec. III, Chap. 8, 9; Sec. V.

[2] H. Hertz, Wiedemann's Annalen, 36, p. 1, 1889.
Die Ausbreitung der elektrischen Kraft, 1892. J. R. Carson, JOUR. A. I. E. E., Oct. 1921, p. 789.

[3] G. Mie, Annalen der Physik, 2, p. 201, 1900.

This raises a new question: If plane traveling waves are only a limiting state, is there not any radiation taking place when the electromagnetic wave is initiated? Or, otherwise expressed, is radiation not an "end effect" with respect to the line? This involves the consideration of the whole surrounding space as the seat of the phenomena, and we need to start from a broader point of view than we did.

Position of the Problem from Maxwell's Point of View—Outline of an Engineering Solution

According to a classification made by Steinmetz, we can say that the transients considered in electrical engineering are either simple time transients, as in machinery, or simple space transients, like voltage and current distribution along a transmission line under steady a-c. conditions, or both time and space transients, as in the case of a line under transient conditions. We have already noticed that the notion of space involved in these transients is unidimensional or depending on one space variable only (equation 1); this supposes implicitly that the electromagnetic field of these transients is in planes. We are led, therefore, to consider more extended classes of transients; three dimensional space transients, as occuring near the ends of a steady a-c. line or near a wireless c. w. antenna and time and three dimensional transients, when steady conditions in such systems are changing. The only means of studying such transients is to resort to Maxwell's equations of the electromagnetic field, connecting the electric force E and magnetic force H. From these equations, follows the same equation for E and H, *viz.*:

$$\frac{\epsilon\,\mu}{c^2}\,\frac{\partial^2 E}{\partial t^2}+\frac{4\,\pi\,\mu\,\gamma}{c^2}\,\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2} \qquad (2)$$

E and H are space vectors, C represents the velocity of light in vacuo, ϵ and μ are the numerical values of the dielectric constant and magnetic permeability of the medium and γ the conductance of the medium expressed in electrostatic units. In air, ϵ and μ are unity and γ zero; in conductors, ϵ is to be taken as zero. From (2) is shown that a disturbance, *i. e.* a local discontinuity of E or H is propagated in the medium at the speed $c : \sqrt{\epsilon\,\mu}$. Equation (2), of which (1) is a particular case, is the "damped wave equation," well known in mathematical physics. Although much work has been done on this equation even in electromagnetics, no solution has been obtained that seems directly available for the present problem, as far as we know. They generally deal with steady state sinusoidal waves along wires, without considering the source producing them, nor how they are initiated. (Sommerfeld, *Ann. Phys.* 67, 1899. Mie, Abraham, *Ann. Phys.* 2, 1900, Hondros. *Ann. Phys.* 1909, etc.) What we want is a knowledge of the whole electromagnetic field proceeding from a disturbance of voltage or current initially localized at a point of the line; the values of the field near the wires will give the amplitude and shape of the traveling wave, the values of the field at great distances from the line will give its radiation.

The mathematical difficulties involved in a general solution are indeed very great, but we believe that the position of the problem as it was sketched is the only one that can lead to a full answer to the question of the attenuation and distortion of traveling waves along wires.

We found it, however, possible to decide whether the radiation is of engineering importance in the attenuation of the waves, either traveling or stationary. By *radiation*, we mean, as usual, the constant limit S towards which tends the flow of energy, as defined by Poynting's theorem, through a sphere enveloping the whole electrical system, the transmission line in the present case, when the radius of the sphere is increasing above any fixed value. The time integral of the preceding limit S will be called the "total radiation" and denoted by W. We will show first that the classical theory of waves (neglecting radiation) gives a "theoretical" distribution of current along the line that is a very close approximation to the "true" distribution, *i. e.*, the unknown distribution taking place when the radiation is considered. Knowing the current in the line, one can calculate the corresponding radiation that must exist; an approximate value will be obtained by substituting the "theoretical" current for the "true" current. If, now, the amount of radiated energy is found to be quite negligible, say one to a hundred, compared to the joulian energy dissipated in the wires, we get an *a posteriori* proof of the rectitude of our premises, that is, that the true distribution of current, is given at, say, one per cent by the classical theory. As attenuation and distortion of the waves are produced by the dissipation of energy in the system, it is logical to conclude that the radiated energy, a cause hundred times less in magnitude than the energy dissipated by joulian heat, will produce a correspondingly small effect in attenuating and distorting the waves, *i. e.*, the radiation will be a negligible factor from an engineering standpoint.

The method is one of successive approximations, as used in experimental physics. It has already been used for calculating the steady radiation of given types of antennas, generally without being substantiated.

(Pierce, El. waves, Chapter IX. F. Cutting, *Proc. I. R. E.*, p. 129, 1922). In the case of a straight oscillator, Abraham found the present method in quantitative agreement with the more elaborate solution based on the full solution of Maxwell's equation. (*Phys. Zeit*, 2, p. 329, 1901). It is to be noticed that in the case of antennas, the radiation is the controlling factor and the joulian dissipation only secondary, while in the case of transmission lines the conditions are reversed. This strengthens our conclusions even more.

We first consider perfect conductors, and later take into account the influence of joulian losses.

When any kind of disturbance of current is propagated along a perfect thin wire, Heaviside shows how the corresponding electromagnetic field is set up by means of spherical traveling waves. (El. Theory, Vol. I, art. 54, etc., where what he calls the "motion of a charge" is entirely equivalent to the propagation of a "current along a wire.") The reflection of current and the corresponding electromagnetic field generated by spherical waves originating at the point of reflection are indicated. (in El. Theory, Vol. II, art. 393, 394.) This theory neglects the diameter of the wires, compared to the length of the disturbance. At that order of approximation, the current (or traveling wave) is propagated undistorted and unattenuated at the speed of light in the air, and reflected as usual, even with regard to possible radiation.

We thus take the distribution of current along the line, either a stationary or traveling sinusoidal one in the steady a-c. case, or a traveling rectangular one in the case of a sudden impulse.

STATIONARY RADIATION FROM A TRANSMISSION LINE

We mean the radiation that takes place under steady a-c. conditions by means of stationary waves, such as the natural oscillations of a line, without joulian losses.

Different modes exist, according to the terminal conditions to which the line is subjected. We consider, as an example, a line free at both ends, thus capable of half-wave length oscillations and multiples thereof.

We assume spherical polar coordinates, with the origin O_1 at the middle of the length l of the wire No. 1, and the polar axis $+ z$ along the same wire.

A length along the wires will be z, r_0 is the distance between a point P and the origin O_1, θ_0 is the angle between $O_1 P$ and the $+ z$ axis; r and θ are the same quantities referred to any point of the wires instead of to O_1; ϕ is the angle (longitude) that the plane $P\, O_1\, z$ makes with the plane of the wires, measured from the direction opposite to the second wire No. 2; d is the distance between the wires. (Fig. 1.)

The distribution of the current along wire 1 is from the preceding:

$$I = i \cdot \cos \frac{\pi n z}{l} \, . \, \sin \frac{\pi n}{l} c t \qquad (3)$$

with $- l/2 \leq z \leq + l/2$, and with reversed sign for the current along wire 2. I is positive, if directed along $+ z$; n is an odd integer giving the "order" of the oscillation; c represents the velocity of light; i is the maximum current.

We have for the electromagnetic field due to the rate of change of a current element $I\, d\, z$, at distances large compared to the wave length of the oscillation, the values first given by Hertz under an equivalent form (*Wied. Ann.* 36, p. 1, 1889; or *Ausbr. Elekt. Kraft.*, p. 153, 1892).

$$E = H = 1/c \frac{\sin \theta}{r} \frac{d}{d\, t} (\hat{I}\, d\, z) \qquad (4)$$

where $I\, d\, z$ is the same as the time derivative of the moment of Hertz's doublet. $\hat{I}$ means I, where the time t has been replaced by $t - r/c$; the current I and magnetic force H are expressed in electromagnetic units; the electric force E is expressed in electrostatic units. At a point P, E lies in the plane of P and the wire, and is perpendicular to r; at the same point, H is perpendicular to this plane. The positive directions are shown on the diagram. We consider the whole transmission line as a large sum of such current elements. At the point P, the elementary $H's$ add themselves, but all the $E's$ do not fall in the same direction. They will do so, however, and will add, provided we recede far enough from the line, because all differences in direction then vanish, *i. e.* θ can be taken as a constant. We can write neglecting quantities of the second order, *i. e.*, $1/r^2$ as compared to $1/r$:

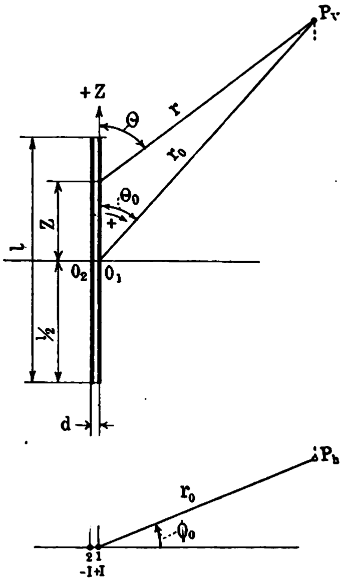

FIG. 1

$$E = H = 1/c \, . \, \frac{\sin \theta_0}{r_0} \, . \, \frac{d}{d\, t} \int_{-l/2}^{+l/2} (\hat{I}_1 - \hat{I}_2)\, d\, z \qquad (5)$$

where I_1 and I_2, the currents in the wires 1 and 2 are to be given their proper time phase $t - r/c$.

At great distances, we have with the same approximation

for wire 1 $\quad r = r_0 - z \cos \theta_0$

for wire 2 $\quad r = r_0 - z \cos \theta_0 + d \sin \theta_0 \cos \phi_0 \qquad (6)$

Still with the same approximation, and neglecting the higher powers of d/l, which is of the order of 10^{-3} to 10^{-5}, (3) gives

$$\dot{I}_1 - \dot{I}_2 = i \,.\, \cos\frac{\pi\, n\, z}{l} \,.\, \cos\frac{\pi\, n}{l}\,(c\,t - r_0 + z\cos\theta_0) \,.\, \frac{\pi\, n\, d}{l}\sin\theta_0\cos\phi_0 \qquad (7)$$

In this expression, z cannot be neglected as compared with r_0, because these quantities figure as argument of the function cos. Substituting in (5), performing the time[4] derivation and the integration with respect to z, we get

$$E = H = (-1)^{\frac{n+1}{2}}\, 2\, i \,.\, \frac{\pi\, n\, d}{l} \,.\, \frac{\sin\frac{\pi\, n}{l}\,(c\,t - r_0)}{r_0}\, \cos\left(\frac{\pi\, n}{2}\cos\theta_0\right)\cos\phi_0 \qquad (8)$$

where n is an odd positive integer, and the signs agree with the assumed convention. This formula represents the steady electromagnetic field from an a-c. transmission line of length l, at a distance r_0 from the origin so large that the ration l/r_0 is very small. The electric and magnetic forces E and H lie on pure spherical waves, traveling at the speed c from the origin on the line. We consider the intersection of the $+\,z$ axis with a sphere as a positive pole (origin of the angles θ) of the sphere. Then, the electric force is directed along meridians and the magnetic force along parallels; the sign in formula (8) is in accordance with the original conventions. (Fig. 2).

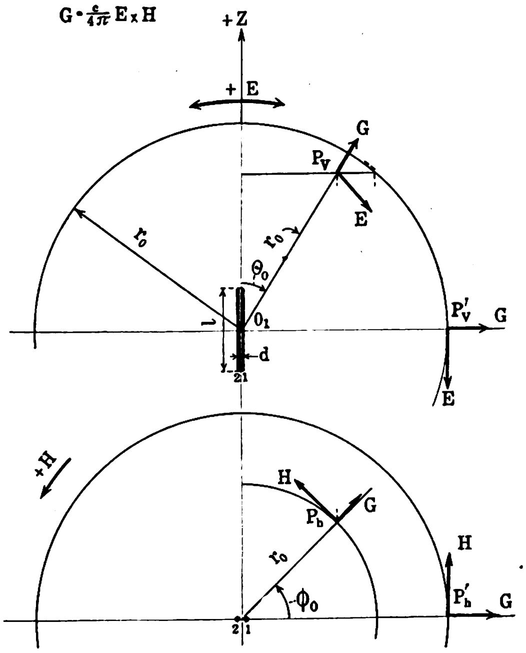

FIG. 2

The main difference between the "field of radiation" of a transmission line and that of a single wire line (antenna), is that, in the first case, the forces are only of the order of d/l, compared to the second case, which is very small.

It is seen that the electric and magnetic forces are perpendicular to the direction of propagation, (the radius of the sphere), as it must be in the case of pure transversal waves; and thus Poynting's vector $E \times H$ is perpendicular to the sphere. Hence, we have for the flow of energy or energy radiated per unit of time

$$S = \frac{c}{4\,\pi}\int E\, H\, d\,\Sigma \qquad (9)$$

integrated over the area of the sphere of radius r_0, with $d\,\Sigma$ as element of area.[5]

Substituting the values (8) for E and H, and performing the integrations, and taking the time average, we obtain:

$$S = C \,.\, \pi^2/2\left(\frac{n\, d}{l}\right)^2 .\, i^2 \quad \text{ergs/sec.} \qquad (10)$$

or

$$= 15 \,.\, \pi^2\left(\frac{n\, d}{l}\right)^2 I^2 = 15\left(\frac{2\,\pi\, d}{\lambda}\right)^2 I^2 \text{ watts} \qquad (11)$$

where I is here the maximum current in amperes and λ the wave length. The *power radiated* by an oscillating transmission line, open at both ends, is thus, for a given current, numerically *proportional to the square of the ratio of the distance between the wires to the wave length of the oscillation.*

We compare the power radiated to the power dissipated by heat, taking as an example the line whose constants are

No. 00 B & S
d = 183 cm.
l = 100 km.
Total d-c. resist. line = 48 ohms
Total external inductance = 0.238 henry

We have, for the power radiated S

for $n = 1$ $\lambda = 2 \times 10^7$ cm.
$S = 0.5 \times 10^{-7}$ watts per max. ampere
f = 1500 cycles/sec.
$n = 2001$ $\lambda \cong 10^4$ cm.
S = 0.2 watts per max. ampere
$f \cong 3 \times 10^6$ cycles/sec.

Due to the sinusoidal distribution of current, the power dissipated by heat is only 48/4 watts per max. ampere at low frequency; it is about 60 times more at three million cycles.

Thus, even at three million cycles, the power radiated by the line is only 1/60 or 1.7 per cent of the power wasted in heat at low frequency, or 1/3600 of the total heat wasted at three million cycles. *Hence, the radiation of oscillating transmission lines is entirely negligible in engineering practise.*

Dr. Steinmetz's formulas amounted to writing the radiated power proportional to $d \,.\, l/\lambda^2$ instead of d^2/λ^2.

Similar results are obtained in the case of forced oscillations of a line or impressed a-c. and load to Carson's formula. (JOUR. A. I. E. E., Oct. 1921, p. 789.)

Instead of considering an ideal sinusoidal current

See note 1.

[5] See Note 2.

distribution along the line, as when there are no losses, we might introduce the actual distribution given by the hyperbolic theory. There would be only more complication in the integrations, but the order of magnitude of the result would remain the same, *i. e.* proportional to d^2/λ^2; thus quite negligible.

TRANSIENT RADIATION FROM A TRANSMISSION LINE

We mean the radiation that accompanies the production of traveling waves. When a disturbance of current is suddenly produced at the end O of a semi-infinite, thin, perfectly conducting wire, it is propagated in first and very close approximation as seen above, unattenuated and undistorted, at the speed of light, along the wire. The wire is supposed to be the only conductor in the empty space. The accompanying electromagnetic field is propagated, at sufficient distance from the source, by spherical waves, also at the speed of light. Its expression at a point P is, when r is great, *i. e.*, such as to make $1/r^2$ negligible compared to $1/r$:

$$E = H = \hat{I}/r \, . \, \frac{1 + \cos\theta}{\sin\theta} \tag{12}$$

r denotes the distance PO; θ is the angle that PO makes with the wire; $\hat{I}$ the current at the origin O, given as a function of time t, where t has been replaced by $t - \frac{r}{c}$: c is the speed of light. I and H are expressed in electromagnetic units; E is in electrostatic units. I is positive when flowing along the positive direction of the wire. The electric force E is directed along meridians; the magnetic force H along parallels.

This formula is substantiated[6] by noticing that it verifies both Maxwell's equation for r large, and that for θ very small, it represents plane electromagnetic waves traveling at the speed c along an infinitely thin wire;

$$E = H = \frac{2\,i}{l} \tag{13}$$

with $l = r \sin\theta$ being the distance of a near by point close to the wire. When the traveling disturbance of current reaches the end of the wire, it can be absorbed by the terminal apparatus without reflection; this is equivalent to starting at the end a new disturbance $-\hat{I}$, of opposite sign, but of same direction of propagation as the first, so that each cancels the other along the wire supposed to be extended. If there is partial or total reflection, a new superimposed disturbance of current $K\hat{I}$ is supposed to be sent along the wire from the end towards the origin, thus propagating in opposite direction, the value and sign of K, which is the coefficient of reflection, depend on the terminal conditions.

We shall consider, just for simplicity, as only disturbance of current the sudden starting or stopping of a constant current at the origin, *i. e.*, "rectangular waves" of current. The stopping will be considered as the starting of a current of opposite sign, in the same direction of propagation.

The case of a *transmission line* reduces itself to the preceding case. We consider a line composed of two parallel semi-infinite wires originating in O_1 and O_2 and at the distance d apart. The coordinates used are the same as before, and are shown on the diagram, Fig. 3. A disturbance is produced at the origin, consisting of a suddenly flowing constant current $+I$ at O_1 and $-I$ at O_2. At *great* distances from the source, the electromagnetic field of the disturbance will be given by the sum of two expressions like (**12**) where due regard is given to the position of the sources O_1 and O_2 and their influence upon the "retardation" $t - \frac{r}{c}$, and also to the sign of the disturbances. This supposes that the presence of one thin wire does not interfere with the electromagnetic field of the other wire at great dis-

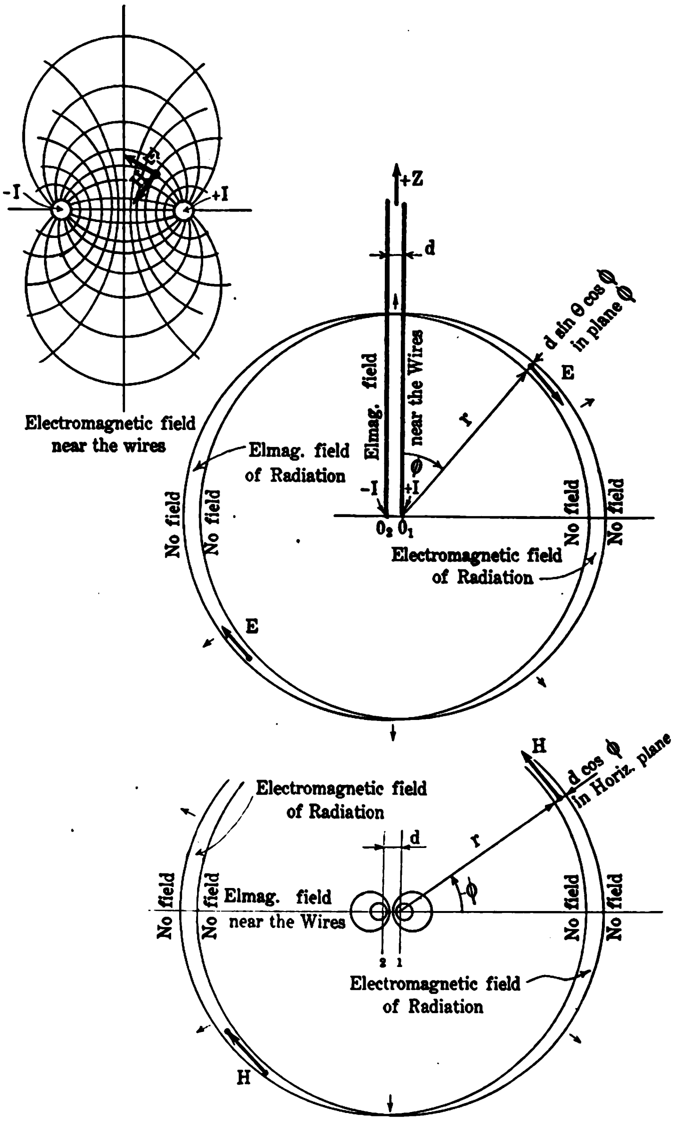

FIG. 3

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[6] See Note 3.

tances; the linearity of the electromagnetic equations permits the addition of the fields.

When a sufficient time has elapsed after the disturbance started at the origin, *i. e.*, when its spherical front will have reached a great distance, in the sense we gave to that expression, the electromagnetic field of the disturbance can be considered as composed of two parts. (See Fig. 3.) One, in the space common to both "front spheres" O_1 and O_2, centered upon the sources O_1 and O_2; the other in the space between the two spherical surfaces, which can be considered as a very thin spherical shell of variable depth

$$d \sin \theta \cos \phi$$

this being equal to the distance $O_1\,O_2$ projected in the direction considered (θ, ϕ). The electromagnetic field outside both spheres is obviously zero. In the shell, it is given by (12), with the proper sign for I, and lasts at a fixed point in the space during the time

$$| \, d/c \sin \theta \cos \phi \, |$$

the bars meaning an absolute value. Inside both spheres the electromagetic field is vanishing, *i. e.*, of the order of $1/r^2$, except near the wires where it is very nearly a plane wave field. It is given there by the difference of two expressions like (**13**)

$$E = H = \frac{2\,\hat{I}}{\rho_1} - \frac{2\,\hat{I}}{\rho_2} \qquad (\mathbf{14})$$

ρ_1 and ρ_2 being the distances of the point close to the wires. This is the well-known value of the electromagnetic field of a plane wave traveling along a perfect two-conductor line, as we considered it first from the classical point of view. Thus, besides the energy stored in the field of the traveling wave near the wires, there is also a small amount of energy present in the indefinitely expanding thin spherical shell of which we spoke, and which the classical theory does not account for. We consider this energy as the *transient radiation from a transmission line,* when acted upon by a sudden applied e. m. f. producing the current I in the line, because this energy is carried to infinity and thus is a net waste for the system. Its amount, calculated in a way entirely similar to that used in the case of the steady radiation above, is[7]

$$W = 3/2\, d \cdot 10^{-9} \cdot I^2 \text{ joules} \qquad (\mathbf{15})$$

where d is the distance between wires in cm. and I the sudden started constant current in amperes.

We consider as an example a *rectangular pulse* of current, *i. e.*, a traveling disturbance of constant current, extending over a length λ cm. The total waste by radiation will be twice as much as given by (**15**), because the same transient radiation occurs when suddenly stopping the current at the origin as when suddenly starting it. The total energy in Joules carried by the pulse along the wires, when it is far enough from the source, is given by $\frac{L \,.\, \lambda \,.\, I^2}{2}$ where L is the self induction in henries per cm. of the line, I the current in amperes. For the line already considered, L is equal to 23.8×10^{-9}. Thus the ratio of the radiated energy to the energy contained in the plane electromagnetic field of the pulse is given by

$$\frac{3\,d}{23.8}\, 1/\lambda = .126\, d/\lambda$$

or, for a pulse extending over a length λ one hundred times the distance d between the wires, only 0.1 per cent of the total energy of the pulse. It is to be noticed that the fundamental frequency of the pulse in this example is as high as 820,000 cycles. Taking as effective resistance the resistance at one million cycles (17.3 ohms per km. of line, Steinmetz loc. at p. 202), it is found that when the pulse travels only one third of its own length, it wastes in heat an amount of energy equal to the energy radiated. This shows how negligible is the effect of radiation.

When a rectangular wave of current reaches the end of a single wire, its stopping generates a new spherical wave, expanding at the velocity of light and centered upon the end R; (Fig. 4) and which is tangent to the first spherical wave originating from O.

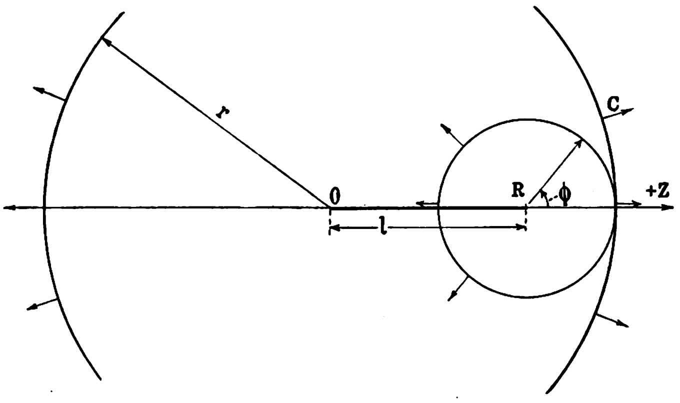

Fig. 4

Similarly, in the case of a two wire line, a spherical shell of electric and magnetic field is generated, centered upon the two ends R_1 and R_2, and tangent to the first shell originating from O_1 and O_2. (Fig. 5.) Thus the total radiation will be the energy carried into infinite space by the two expanding spherical shells. As the two shells have the part $X_1\,X_2$ partly in common, where their fields add to each other, the total energy radiated will not be exactly that obtained by adding the energies of both shells separately, but will be a little less. It is shown that taking this sum an error is committed that will not exceed the ratio of the angle $\frac{X_1\,X_2}{r}$ to 2, *i. e.*, of the order of d/l,[8] or between 10^{-3} to 10^{-5} or less. Hence, to that approximation, we can say that *the starting current and successive reflections will each add a given amount to the energy lost by radia-*

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[7] See note 4.

[8] See note 6.

tion. The solution of specific cases is thus directly given by the preceding. We give the following results as illustrations of the method.

1. The *sudden* establishment from one end of a line of a sustained continuous current I, without reflection at the other end, wastes

$$3\,d\,10^{-9}\,I^2 \text{ joules} \tag{16}$$

by radiation; this result is independent of the length of the line or of its material; it depends only on the distance d between the two wires of the line. In this formula and in all the following, d is in centimeters and I in amperes.

Taking the same 100 km. transmission line as previously, and assuming only the d-c. resistance even during the transient conditions, it is seen that the energy wasted by radiation is only 1/10,000th of the energy wasted in heat during the time in which the line is being charged, and the steady d-c. conditions are reached. For a 1-km. line, the radiated energy would still be only one per cent of the energy wasted in heat during the transient conditions. The same remark as to the order of magnitude of the radiation compared to the joulian losses applies to all following examples.

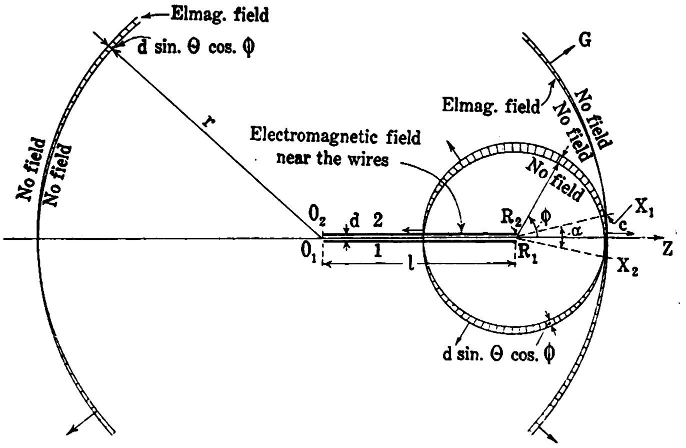

FIG. 5

2. A total *reflection* of a "rectangular wave" of current I at an *open* end of a line radiates[9]

$$4\,d\,10^{-9}\,I^2 \text{ joules;} \tag{17}$$

at a *grounded* end,

$$2\,d\,10^{-9}\,I^2 \text{ joules.} \tag{18}$$

In general, we have a *partial reflection* when the amount of current $K\,I$, (in magnitude and sign), is reflected and thus the rest of the wave is absorbed in the terminal apparatus. The radiated energy there is given by

$$3/2\,(1 - 2/3\,K + K^2)\,d\,10^{-9}\,I^2 \text{ joules} \tag{19}$$

of which the three preceding formulas are particular cases. ($K = 0, K = -1, K = 1$)

3. A complete transposition, *i. e.* a sudden rotation of 180 deg. of the plane of the wires of the line, produces a radiation of

$$6\,d\,10^{-9}\,I^2 \text{ joules,} \tag{20}$$

or four times as much as given by formula (**15**), because a transposition amounts to a sudden stopping of the current plus a sudden starting of the same current with opposite sign, in the same direction of propagation or to a sudden starting of twice the same current in the same direction, with opposite sign.

Hitherto, we considered a current flowing suddenly in the line at maximum value or a "rectangular wave" as we called it. Any other law for the current as a function of the time at the origin, as for example a sinusoidal law, can be assumed, with only changes in the calculations, but yielding the same conclusions as to the magnitude of the radiation effect.

We also neglected the influence of the ground and supposed a transmission line isolated in space. A perfectly conducting ground can be accounted for by taking with the line its image with respect to the ground; the resulting radiation is obviously less.

If, instead of considering disturbances between the two wires of a line, we consider disturbances between line and ground, the distance d between the wires would have to be replaced by twice the height of the line above the ground, in the case of a good conducting ground. This can magnify the radiation effect about twenty times, but keeps it still negligible compared to the effect of heat dissipation in the conductors.

CONCLUSIONS

The present conclusions settle the question of the relative unimportance of radiation in the attenuation and distortion of traveling waves along lines. They, however, leave the way open for the investigation of other more important causes of attenuation and distortion, among which we suggest the transient penetration of the current into the wire at the front of the disturbance, as mentioned above; the dissipation of energy in the ground due to the fact that some lines of force of a two-wire transmission line reach the ground and cause an attenuation as in the case of wireless waves; the non-homogeneity of the transmission line for high-frequency waves, *i. e.*, the slight periodic variations that occur in the constants of the line due to presence of poles and curved spans. The influence of changes in direction in the line, as far as radiation is concerned, can be found by the method employed here and is negligible.

SUMMARIZING

1. A criticism of the classical theory of propagation of traveling waves along a two-wire transmission line has been given showing that, as far as the shape of the wave near the front is concerned, this theory is still unsatisfactory.

2. Dr. Steinmetz's theory of the effect of radiation on traveling waves has been examined.

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[9] See note 5.

3. The position of the problem of radiation of lines has been sketched from Maxwell's electromagnetic point of view, in connection with the accepted engineering theories of transients.

4. An engineering solution of the problem is offered, showing that under steady a-c. as well as under transient conditions, *the effect of radiation upon electromagnetic waves propagating along transmission lines is quite negligible as compared to the effect of heat dissipation in the wires.*

The procedure used here in the case of steady a-c. conditions is to be considered an extension of a method due to Abraham;[10] it is different in the case of transients, and thought to be new.

5. Other possible causes of attenuation and distortion of traveling waves are suggested.

NOTE 1

The integration to be performed is obtained by substituting the time derivative of $\hat{I}_1 - \hat{I}_2$, given by (7) into (5); so

$$E = H = -\imath\frac{\pi n}{l} \cdot \frac{\sin\theta_0}{r_0} \cdot \frac{\pi n d}{l} \cdot \sin\theta_0$$
$$\cdot \cos\phi_0 \int_{-l/2}^{+l/2} \cos\frac{\pi n z}{l} \cdot \sin\frac{\pi n}{l}(c\,t - r_0 + z\cos\theta_0) \cdot d\,z$$

But

$$\sin\frac{\pi n}{l}(c\,t - r_0 + z\cos\theta_0) = \sin\frac{\pi n}{l}(c\,t - r_0) \cdot \cos\frac{\pi n}{l}(z\cos\theta_0) + \cos\frac{\pi n}{l}(c\,t - r_0) \cdot \sin\frac{\pi n}{l}(z\cos\theta_0)$$

The second term contributes nothing to the integration, for $\sin\frac{\pi n}{l}(z\cos\theta_0)$ changes its sign with z. We thus have only to perform the integration

$$\frac{\pi n}{l}\int_{-l/2}^{+l/2} \cos\frac{\pi n z}{l} \cdot \cos\frac{\pi n}{l}(z\cos\theta_0)\,d\,z$$
$$= \int_{-\frac{n\pi}{2}}^{+\frac{n\pi}{2}} \cos x \cdot \cos a\,x \cdot d\,x$$
$$= 1/2\int_{-\frac{n\pi}{2}}^{+\frac{n\pi}{2}} \cos(1+a)\,x \cdot d\,x + 1/2\int_{-\frac{n\pi}{2}}^{+\frac{n\pi}{2}} \cos(1-a)\,x \cdot d\,x$$
$$= \frac{\sin(1+a)\,n\,\pi/2}{1+a} + \frac{\sin(1-a)\,n\,\pi/2}{1-a}$$
$$= \frac{2}{1-a^2} \cdot \sin\frac{\pi n}{2} \cdot \cos a\frac{\pi n}{2}\ (n\text{ odd})$$
$$= \frac{2}{\sin^2\theta_0} \cdot (-1)^{\frac{n-1}{2}} \cdot \cos\left(\frac{\pi n}{2}\cos\theta_0\right)$$

which, multiplied by the constant factors that we did not write through, gives formula (8).

NOTE 2.

The flow of energy S, given by (9), is

$$S = \frac{c}{4\pi}\int E\,H\,d\,\Sigma$$

integrated over the area of the sphere of radius r_0; the values of E and H are to be taken from formula (8), and $d\,\Sigma$ is the element of area in spherical polar coordinate, *i. e.*,

$$r_0^2 \sin\theta_0\,d\,\theta_0\,d\,\phi_0$$

We thus have to integrate

$$S = \frac{c}{4\pi}\imath^2\left(\frac{2\pi n d}{l}\right)^2 \frac{\sin^2\frac{\pi n}{l}(c\,t - r_0)}{r_0^2}$$
$$\int_{\theta_0=0}^{\pi}\int_{\phi_0=0}^{2\pi} \cos^2\left(\frac{\pi n}{2}\cos\theta\right)\cos^2\theta_o \sin\theta_o\,r^2_o\,d\theta_o\,d\phi_o$$

But we have, dropping unnecessary indices,

$$\int_0^{2\pi} \cos^2\phi \cdot d\,\phi = \pi$$

and

$$\int_0^{\pi} \cos^2\left(\frac{n\pi}{2}\cos\theta\right)\sin\theta\,d\,\theta = 1$$

The double integral thus reduces to $n\,r_0^2$ and the value of S becomes

$$S = c\,i^2\left(\frac{\pi n d}{l}\right)^2 \sin^2\frac{\pi n}{l}(c\,t - r_0)$$

of which the time average is given by replacing $\sin^2$ by 1/2. Hence formula (10)

NOTE 3.

The equations (12) for E and H can be substantiated by proving that they satisfy both Maxwell's equations for r large, *i. e.* when $1/r^2$ is negligible compared to $1/r$. Maxwell's equations are, in air

$$1/c\frac{\partial E}{\partial t} = \text{curl}\,H$$
$$-1/c\frac{\partial H}{\partial t} = \text{curl}\,E$$

where E and H are space-vectors, E being expressed in electrostatic units and H in electromagnetic units.

The component of the curl of a vector, at a given point, in a given direction is defined as the line integral of that vector taken around a unit area located at the given point, of which the positive normal coincides with the given direction.

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[10] *Phys. Zeitschrift*, *2*, p. 239, 1901.

The curl components of E and H in the direction of the radius (see Fig. 3) are seen to be zero or of the second order, (*i. e.*, of the order of $1/r^2$), thus negligible. The components of curl E along a meridian line, and of curl H along a parallel line are obviously zero, because the vector is then perpendicular to the line of integration.

The component of curl H in the positive direction of a meridian line, (*i. e.*, towards the increasing θ), is given by

$$\text{curl}_\theta \, H =$$

$$\frac{-H(r+\Delta r) \, . \, (r+\Delta r)\sin\theta \, . \, \Delta\theta + H(r) \, . \, r \, . \sin\theta \, . \, \Delta\theta}{\Delta r \, . \, r\sin\theta\,\Delta\theta}$$

where $H\,(r)$ means the scalar value of H at the distance r. But we have, neglecting quantities of the second order, and using the value of H given by (1):

$$H\,(r + \Delta r) = H\,(r) + \frac{\dfrac{\partial \hat{I}}{\partial r}}{r} \, . \, \frac{1+\cos\theta}{\sin\theta} \, . \, \Delta r$$

$$= H\,(r) - 1/c \, \frac{\partial E}{\partial t} \, . \, \Delta r$$

Substituting in $\text{curl}_\theta H$, at the same approximation, we find the first equation of Maxwell.

The same procedure with regard to the signs leads to the second equation of Maxwell.

NOTE 4.

Formula (**15**) represents the energy carried to infinity by the spherical shell of variable depth

$$d \sin\theta\cos\phi$$

in which the value of the electromagnetic field is given, for $-\pi/2 \leqq \phi \leqq +\pi/2$, by formula (**12**)

$$E = H = \hat{I}/2 \, . \, \frac{1+\cos\theta}{\sin\theta}$$

where I is the constant value of the suddenly impressed current. For $\pi \; 7 \; Z/\phi \angle \frac{3\,\pi}{2}$, the sign of both E and H is to be changed.

At a given point on the sphere of (large) radius r, this electromagnetic field lasts during the time

$$|\, d/c \sin\theta\cos\phi \,| \qquad \text{(}c\text{ speed of light)}$$

the bars meaning a quantity that is positive.

The total energy radiated W will be given by the time integral extended over the sphere of radius r of Poyting's vector, which is directed everywhere along the radius outwards the sphere; *i. e.*

$$\frac{c}{4\,\pi} E \times H \,|\, d/c \sin\theta\cos\phi$$

integrated over the sphere of which the element of area is

$$r^2 \sin\theta \, d\,\theta \, d\,\phi$$

We thus have to perform

$$W = \frac{c}{4\,\pi} I^2/r^2 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left[\frac{1+\cos\theta}{\sin\theta} \right]^2 .$$

$$|\, d/c \sin\theta\cos\phi \,|\, r^2 \sin\theta \, d\,\theta \, d\,\phi$$

But we have

$$\int_0^{2\pi} |\cos\phi| \, d\,\phi = 4$$

and

$$\int_0^{\pi} [1+\cos\theta]^2 \, d\,\theta = \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\,\theta$$

$$= \pi + 0 + \pi/2 = \frac{3\,\pi}{2}$$

The value of W is thus

$$W = 3/2 \, d \, I^2 \qquad \text{elmag C. G. S. units}$$

which is equivalent to (**15**).

NOTE 5.

The total reflection "with change of sign" of a rectangular wave of current at the free end of a line can be considered as the result of the stopping of the incoming wave $+I$ at the end, by starting at the end a new wave $-I$ along the same direction of propagation as that of the incoming wave, plus starting a new wave $-I$ in the opposite direction, *i. e.*, back into the line. This last wave is the wave "reflected with change of sign." The sign of a wave bears no relation at all to the direction of propagation.

The reflection at a free end of a line is thus equivalent to starting simultaneously two waves $-I$ from the end in two opposite directions. The electromagnetic field of such a double disturbance along a single wire is easily deduced from that given in formula (**12**), for a single disturbance. It is, with regard to the signs

$$E = H = \frac{(-\hat{I})}{r} \frac{1+\cos\theta}{\sin\theta} + \frac{(-\hat{I})}{r} \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{(-2\,\hat{I})}{r} \frac{1}{\sin\theta}$$

$(-I)$ is the current started in both directions at the free end, equal in absolute value to the incoming current, and where the time t, of which I is a function in the general case, has been replaced by $t - r/c$, r being taken from the free end.

The case of a two-wire (transmission) line reduces to the preceding one as was seen. Thus, in the case of a rectangular wave of current along a line, the electromagnetic field due to the sudden starting at the end of the line of a current ($-I$) in both directions is:

$$E = H = -\frac{2\hat{I}}{r}\frac{1}{\sin\theta}$$

This field exists in a spherical shell of variable depth, $d \sin\theta\cos\phi$, expanding at the speed of light, as it has been shown. It lasts at a fixed point in space during the time

$$| d/c \sin\theta\cos\phi |$$

the bars meaning a positive quantity. The amount of radiated energy W can be found exactly as in the preceding note. We have:

$$W = \frac{c}{4\pi}\frac{4I^2}{r^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{1}{\sin^2\theta}\,| d/c \sin\theta\cos\phi |\; r^2 \sin\theta\, d\theta\, d\phi$$

what reduces immediately to:

$$W = 4\,d\,I^2 \qquad \text{elmag c. g. s. units}$$

which is equivalent to formula (**17**).

In general, when the amount of current $K I$ (in sign) is reflected into the line, the rest being absorbed in the terminal apparatus, we have for the electromagnetic field, with regard to the sign,

$$E = H = \frac{-\hat{I}}{r}\cdot\frac{1+\cos\theta}{\sin\theta} + \frac{(K\hat{I})}{2}\cdot\frac{1-\cos\theta}{\sin\theta},$$

which exists in a spherical shell of variable depth.

$$d \sin\theta\cos\phi$$

expanding at the speed of light.
The radiated energy is, as above,

$$W = \frac{c}{4\pi} I^2/r^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left[\frac{(1-K)+(1+K)\cos\theta}{\sin\theta}\right]^2 \cdot | d/c \sin\theta\cos\phi |\, r^2 \sin\theta\, d\theta\, d\phi$$

The value for W follows

$$W = 3/2\,(1 - 2/3\,K + K^2)\,d\,I^2 \text{ elmag. c. g. s. units}$$

which is equivalent to (**19**).

NOTE 6.

The position of the point X_1, limiting the length $X_1 X_2$ (Fig. 5) along which the two spherical shells can have parts in common is determined by the intersection of a circle of radius r centered upon O_1, and of a circle of radius $r - l$ centered upon R_2, r being very large compared to l. This allows to approximate the circles by parabolas of the same curvature, *i. e.*, of which the parameter p is equal to the radius of the circles; the axis of the first parabola will coincide with the wire $O_1 R_1$, the axis of the second with the wire $O_2 R_2$ We so have

$$y - d = \sqrt{2(r-l)x} \qquad \text{Par. 2}$$
$$y = \sqrt{2rx} \qquad \text{Par. 1}$$

with obvious axis of coordinates. Calling ($\xi_1 \eta$) the coordinates of the point X_1, we deduce

$$\eta = \sqrt{\frac{r}{r-l}}\cdot\sqrt{2(r-l)}\,\xi = \sqrt{\frac{r}{r-l}}\,(\eta - d)$$

As ξ is very small, η is the distance of X_1 to wire 1—We have thus, from the preceding equation, neglecting the square and higher powers of r/l:

$$\eta/r = 2\,d/l\,.$$

which proves that, viewed from a point of the line, the length $X_1 X_2$ does not cover an angle larger than

$\frac{4d}{l}$, provided l is small compared to r.

Hence, calculating the energy radiated by the two spherical shells in contact as if they were not in contact, *i. e.*, integrating over π instead of π minus the angle η/r, will introduce only an error of the order of d/l, thus quite negligible.

LIST OF RESEARCH BULLETINS—Continued

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17. Electromagnetic Theory of the Telephone Receiver, with Special Reference to Motional Impedance. A. E. Kennelly and H. Nukiyama; March, 1919.
18. A Rectangular-Component Two-Dimensional Alternating-Current Potentiometer. A. E. Kennelly and Edy Velander; July, 1919.
19. Current Distribution in Armature Conductors. Waldo V. Lyon; July, 1919.
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24. The Transient Process of Establishing a Steadily Alternating Current on a Long Line, from Laboratory Measurements on an Artificial Line. A. E. Kennelly and U. Nabeshima; January, 1921.
25. A Dynamometrical Comparator. A Differential Dynamometer for Accurate Comparison of Alternating and Direct-Current Strengths. Edy Velander, February, 1921.
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27. Acoustic Impedance and Its Measurement. A. E. Kennelly and K. Kurokawa, February, 1921.
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29. A Mechanical Frequency-Meter of Telephonic Range. A. E. Kennelly and C. H. Manneback, January, 1922.
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Dissymetrical Electrical Networks

by
A. E. Kennelly

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LIST OF RESEARCH BULLETINS

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*1. The Economical Transportation of Merchandise in Metropolitan Districts. H. Pender and H. F. Thomson; March, 1912.

*2. Notes on the Cost of Motor Trucking. H. Pender and H. F. Thomson; October, 1912.

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*Out of print.

Dissymmetrical Electrical Conducting Networks

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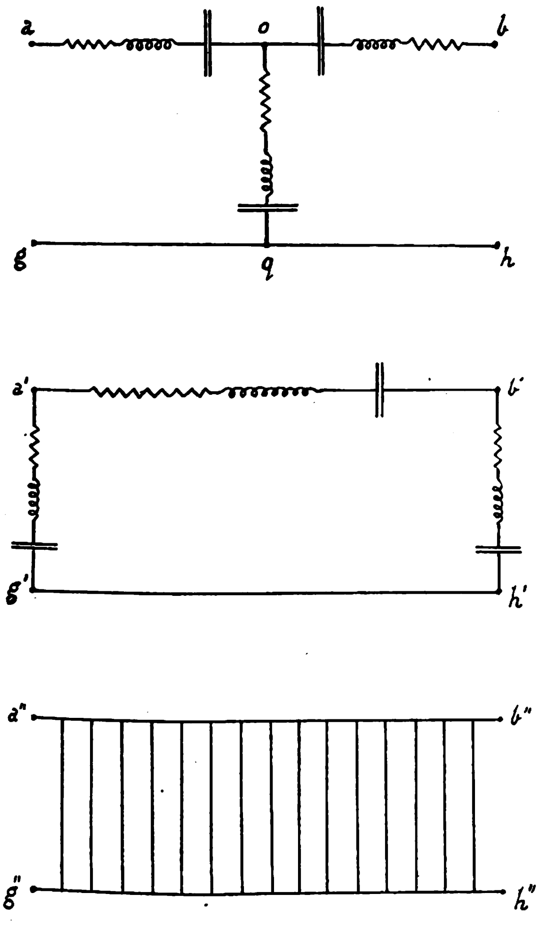
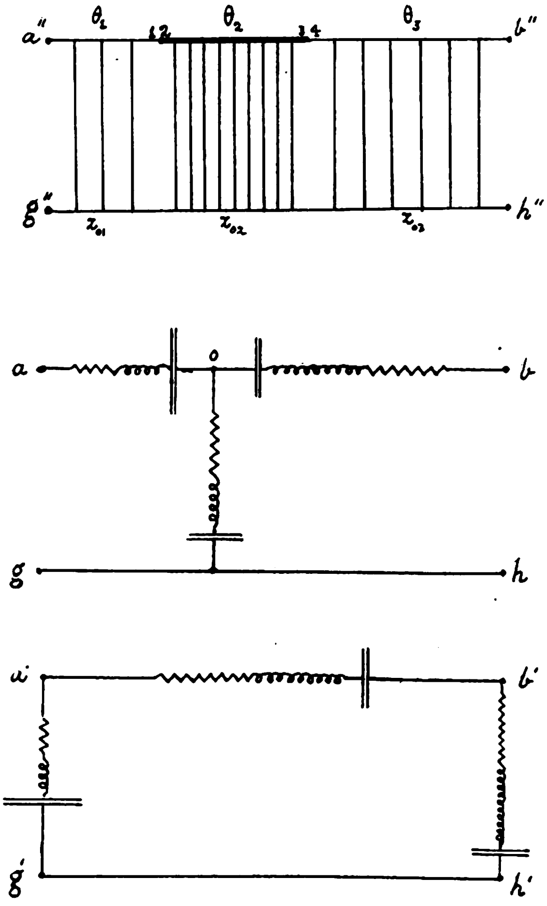
IT is well known that a symmetrical *T*-connection of three simple impedances *a o*, *o b*, and *o q*, Fig. 1, in which the two arms *a o*, and *b o* are electrically equal, and with their common terminal *o* connected to the ground or return conductor *g h* through the impedance *o q*, can be completely replaced,* at any single assigned alternating-current frequency, by a certain symmetrical π, $a' b' g' h'$, Fig. 2, in which the two pillars $a' g'$ and $b' h'$, have equal impedances; also that either of these two symmetrical systems can be completely replaced, at the same frequency,† by a single smooth line conductor $a'' b''$, Fig. 3, having uniformly distributed series impedance in its conductor and also uniformly distributed lateral leak admittance between the conductor and the return conductor $g'' h''$. Such a line conductor may be regarded as subtending or possessing, at the frequency in question, a certain complex hyperbolic angle θ, and also a certain single surge impedance or "characteristic impedance," z_0 ohms. Consequently, both the symmetrical *T* conducting system of Fig. 1, and the symmetrical π of Fig. 2, may be regarded as possessing the same complex angle θ and surge impedance z_0, by virtue of equivalence. In any network of conductors, carrying alternating currents of the assigned frequency in the steady state, any of the three equivalent systems of Figs. 1, 2 and 3 may be interchanged, without disturbing the distributions of potential, current and power, at and outside the terminals of the system.



FIGS. 1, 2 AND 3—SYMMETRICAL *T*, π AND SMOOTH LINE, EQUIVALENT AND INTERCHANGEABLE



FIGS. 4, 5 and 6—COMPOSITE LINE OF THREE SECTIONS WITH ITS EQUIVALENT DISSYMMETRICAL *T* AND π

Moreover, if two or more simple smooth line conductors, of the type indicated in Fig. 3, are connected in series to form a composite line as in Fig. 4, it is known that the composite line system $a'' b'' g'' h''$ (Fig. 4) can be replaced by its equivalent *T* of Fig. 5,‡ or by its equivalent π of Fig. 6, and that these equivalent conductors are, in general, dissymmetrical. For a given composite-line system of Fig. 4 operated in the steady state at a single frequency, there can be only one equivalent *T* and also only one equivalent π. On the other hand, neither the equivalent *T* of Fig. 5, nor the equiva-

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* Bibliography (1).

† Bibliography (3).

‡ Bibliography (4).

lent π of Fig. 6 can determine the elements of smooth line section in the equivalent composite-line system of Fig. 4. In the absence of specific information concerning the number and characters of the component-line sections, an infinite number of different composite-line systems like that of Fig. 4, could be found as equivalents for the dissymmetrical T or π of Figs. 5 and 6. The question thus arises as to what is the simplest smooth-line system which can replace a given dissymmetrical T and its corresponding dissymmetrical π?

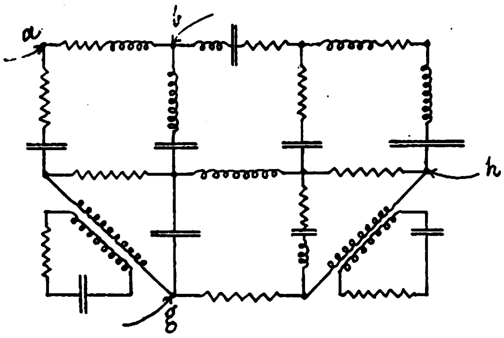


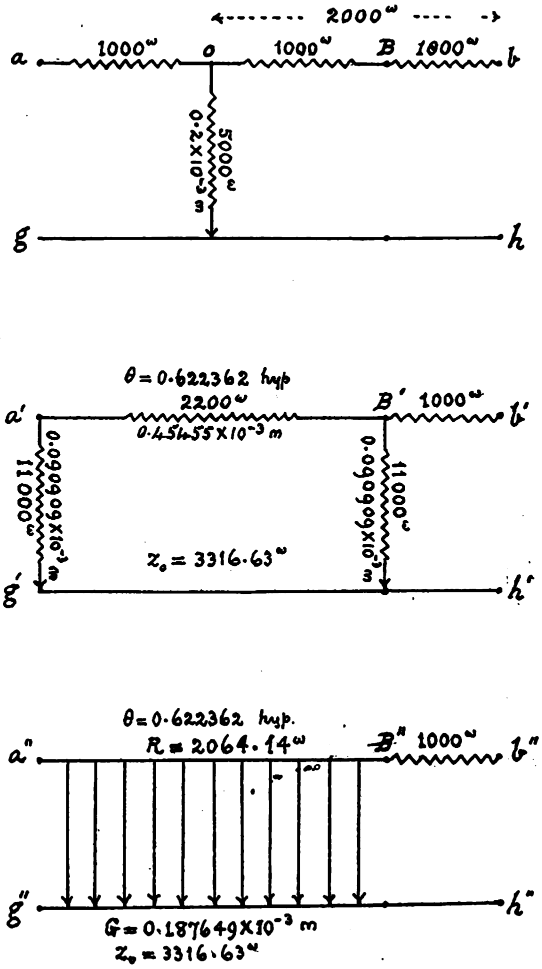
Fig. 7—Network of Impedances Replaceable by One Dissymmetrical T or π with Respect to the Two Pairs of Terminals, *a*, *g*, and *b*, *h*.

Again, if any network of conductors, such as that of Fig. 7, carrying alternating currents of a single frequency, and two pairs of terminals are selected arbitrarily from the system, such as *a*, *g*, and *b*, *h*; then the system behaves with respect to these pairs of terminals like a certain equivalent T of Fig. 5, and also like the corresponding equivalent π of Fig. 6, these two systems being mutually equivalent and, in general, dissymmetrical. The question then also arises as to what is the simplest smooth-line conductor system which can be regarded as replacing the network, with respect to the pairs of terminals *a*, *g* and *b*, *h*?

A symmetrical network of conductors, with respect to two pairs of terminals, may be defined as one whose equivalent T and π are symmetrical, like Figs. 1 and 2. A dissymmetrical network is then one which, with respect to the two pairs of terminals, has a dissymmetrical equivalent T and π, like Figs. 5 and 6. In general, networks are dissymmetrical, and symmetrical networks are particular cases.

Although we always discuss a network with reference to two pairs of terminals, such as *a*, *g* and *b*, *h* in Fig. 7; yet the number of terminals involved may be reduced, in particular cases, from four to three, by the merging of two terminals. Thus, we might select the two pairs of terminals, *a*, *g* and *b*, *g*, which employ the terminal *g* in common.

As has already been pointed out, a network which is symmetrical with respect to two pairs of terminals becomes defined either by a certain symmetrical T and π, or by a certain hyperbolic angle θ and accompanying surge impedance z_0. We can then readily compute what changes in alternating-current potential difference, current and power will be brought about at one of the pairs of terminals by any assigned electric change steadily impressed on the other pair. When, however, the network is dissymmetrical, we shall find that it requires for its definition one additional characteristic. A convenient additional characteristic is the "inequality ratio" q; so that the network acquires three characteristics θ, z_{ab} and q, which are capable of being measured in any given case, by following a suitable technique at each pair of terminals. Having ascertained these three determining characteristics for the network, we can either draw immediately its equivalent T and π; or we can compute what change in potential difference, current and power will be produced at one pair of terminals, say *a*, *g*, by a given change established steadily at the other pair *b*, *h*. The computation is effected by following the same process as in the symmetrical case; but with a slight modification.



Figs. 8, 9 and 10—Dissymmetrical T Reduced to Symmetry by the Retention of a Permanent Series Load at the *b* Terminal with the Correspondingly Loaded π and Smooth Line.

The subject can best be studied with reference to continuous-current lines and networks; *i. e.*, with reference to the frequency of zero. The results so obtained are immediately applicable to alternating-current systems, by the substitution of complex numbers for real numbers, in the well known way.

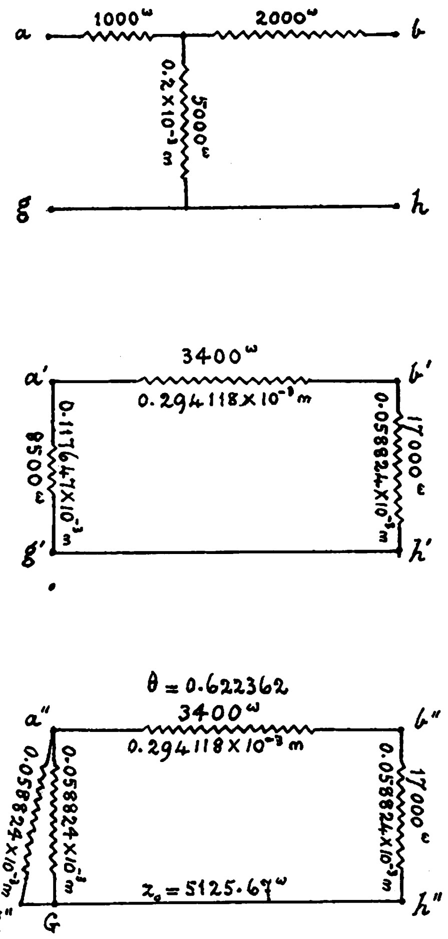
Fig. 8 shows a simple dissymmetrical *T*. In this particular case the branch *a o* has 1000 ohms and the branch *b o* 2000 ohms. Two methods suggest themselves for dealing with it; namely, (1) by reducing it to a terminally loaded symmetrical *T*, and (2) by retaining the dissymmetry, but making measurements from each end in turn. We may consider these two methods successively.

(1) *Method of Reduction to Symmetry.* The simplest way to reduce the *T a, o, b* of Fig. 8 to symmetry, is perhaps to cut off the portion *B, b* from the preponderating arm, so as to leave the remainder *o, B* equal to the branch *a, o*. We thus obtain symmetrical *T a, o, B* with a load *B, b*, in this case 1000 ohms, permanently attached to the terminal *B*. The corresponding loaded symmetrical π is shown in Fig. 9; where the same load *B′, b′*, is permanently attached to the *B′* terminal. Fig. 10 gives the terminally loaded smooth line *a″ B″*, corresponding to the combinations of Figs. 8 and 9. It has an angle of 0.622362 hyp., a surge resistance of $z_0 = 3316.63$ ohms, a total conductor resistance R of 2064.14 ohms and a total leakance G of 0.187649 millimho. The same load of *B″ b″* = 1000 ohms is permanently attached to the *B″* terminal.

It is readily possible to substitute for the dissymmetrical *T a, o, b*, of Fig. 8, the terminally loaded smooth line of Fig. 10, in any steady-state system. The smooth line is readily dealt with in computation, and the permanent load *B″ b″* merely modifies the conditions existing at *B*, in a manner deducible directly from Ohm's law. If preferred, the load may be dealt with by assigning to it an auxiliary angle θ'; so that the position angle at *a*, becomes $\theta + \theta'$.

Although the reduction to symmetry effected by reducing the higher-impedance branch of the *T* to equality with the lower-impedance branch, is perhaps the simplest, especially in a continuous-current case like that of Fig. 8; yet it is by no means the only method of procedure open to use. Thus, the dissymmetrical *T* of Fig. 8, might be replaced by its corresponding dissymmetrical equivalent π, as in Figs. 11 and 12. The π, *a′ b′ g′ h′* of Fig. 12 may be reduced to symmetry, by cutting off part of the leak *a′ g′* as a terminal load, and leaving the remainder at *a″ G*, Fig. 13, equal to the leak *b″ h″*. This leaves the symmetrical π, *a″ b″ G h″*, of angle $\theta = 0.622362$ and surge impedance $z_0 = 5125.67$ ohms, but with the leak *a″ g″* of 0.058824 millimho permanently attached to *a″*. This symmetrical π might now be replaced either by a leak-loaded symmetrical *T*, or by a smooth line with the same terminal leak load.

Again it would be possible to reduce the *T* of Figs. 8 and 11 to symmetry, by employing permanent pairs of unequal terminal series loads, as for instance 100 ohms at *a*, and 1100 ohms at *b*, so as to produce a symmetrical *T* with 900 ohms in each branch. This could be done in an infinite number of ways, each giving a different θ, a different z_0, and a different pair of permanent terminal loads. It would also be possible to reduce the π of Fig. 12 to symmetry, by employing pairs of unequal terminal leak loads. There would be an infinite number of such possible pairs, with corresponding values of θ and z_0. Consequently, a dissymmetrical network admits of being reduced to symmetry for computation, in one way with a single terminal series load, in one way with a single terminal leak load, in an infinite number of ways with a pair of opposite terminal series loads, and also in an infinite number of ways with a pair of opposite terminal leak loads. In practise, the two single-load methods are preferable and easier to use.



Figs. 11, 12 and 13. Dissymmetrical *T* Replaced by Its Correspondingly Dissymmetrical π, Which is Reduced to Symmetry by the Retention of a Permanent Leak at the *a″* Terminal.

(2) *The Method of Applying the Inequality Ratio q.* In this method, the system is left in its dissymmetrical state; but measurements are made from each pair of terminals in turn, to determine the apparent surge impedance from each end.

Returning to the dissymmetrical *T* of Figs. 8 and 11, we can measure the resistance R_{af} offered between the terminal *a* and the return conductor or ground *g*, when the system is freed or opened at the opposite terminals b, h. We may also measure the resistance R_{ag} offered between the same terminals at a, when the end b is grounded, or the terminals b, h are short-circuited. Following the usual rule for finding the angle of a symmetrical smooth line, the apparent angle of the system is expressed by:

$$\tanh \theta = \sqrt{R_{ag}/R_{af}} \qquad \text{numeric } \angle \quad (1)$$

In the particular case of Fig. 8, the resistance R_{ag} would be 2428.57 ohms, and $R_{af} = 6000$ ohms, from which $\tanh \theta = 0.636209$ and $\theta = 0.751779$ hyp. radians. Similarly, according to the regular rule for a symmetrical line, the apparent surge impedance z_{0a} from the end a, is

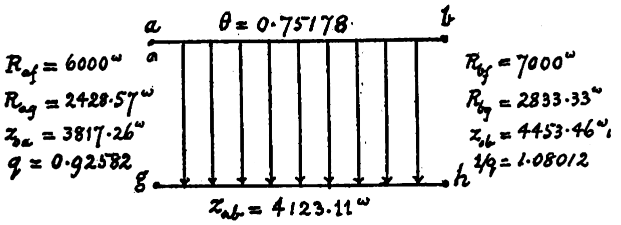


FIG. 14—NETWORK OF FIG. 11 WITH SINGLE ANGLE θ, BUT WITH SURGE IMPEDANCES z_{0a} AND z_{0b} AS DETERMINED FROM EACH END.

$$z_{0a} = \sqrt{R_{af} \,.\, R_{ag}} \qquad \text{ohms } \angle \quad (2)$$

which in the case considered, is 3817.26 ohms. If now we repeat the measurements from the b end, successively freeing and grounding the a terminal, we find R_{bf} and R_{bg}, which would be 7000 and 2833.33 ohms respectively. As before,

$$\tanh \theta = \sqrt{R_{bg}/R_{bf}} = 0.636209 \qquad \text{numeric } \angle \quad (3)$$

whence $\theta = 0.751779$. This is the same value as was found by (1), from the measurements at a. We may express this relation as a general rule thus: If in any dissymmetrical network of conductors, such as that of Fig. 7, we take two pairs of terminals like a, g, and b, h, and measure, at an assigned frequency, the impedance offered by the network between each pair, with the opposite pair first opened and then shorted, we may denote these impedances by R_{af}, R_{ag} and R_{bf}, R_{bg}, ohms ∠ respectively. Then the ratios R_{ag}/R_{af} and R_{bg}/R_{bf} will be equal to each other, and to the square of the tangent of the angle subtended by the network with respect to those four terminals.

On the other hand, if the network is dissymmetrical, the apparent surge impedance from b, or

$$z_{0b} = \sqrt{R_{bf} \,.\, R_{bg}} \qquad \text{ohms } \angle \quad (4)$$

will not be the same as z_{0a}. In the case considered, $z_{0b} = 4453.46$ ohms. The geometrical mean z_{ab} of the two opposite apparent surge impedances is defined by

$$z_{ab} = \sqrt{z_{0a} \,.\, z_{0b}} \qquad \text{ohms } \angle \quad (5)$$

is a characteristic property of the network, in this case 4123.11 ohms. It may be called the "geomean surge impedance" of the network.

The ratio of the a-end surge impedance z_{0a} to the geomean surge impedance z_{ab} may be called q, the "inequality ratio" of the system. It will be defined by:

$$q = z_{0a}/z_{ab} = \sqrt{R_{ag}/R_{bg}} = \sqrt{R_{af}/R_{bf}} = \sqrt{z_{0a}/z_{0b}}$$

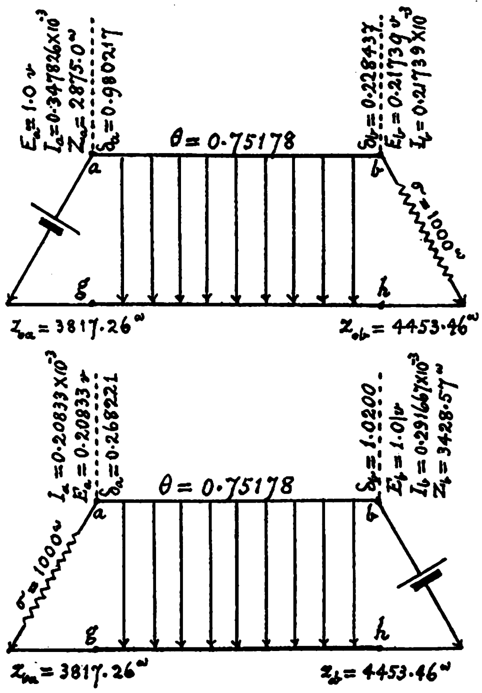
$$= \sqrt{\frac{\rho_1 + \mathcal{R}}{\rho_2 + \mathcal{R}}} = \sqrt{\frac{g_2 + \nu}{g_1 + \nu}} \qquad \text{numeric } \angle \quad (6)$$

In this case the inequality ratio is $q = 0.92582$.

We may thus represent either the dissymmetrical T of Fig. 11, or its corresponding π of Fig. 12, by the dissymmetrical line system of Fig. 14. Here the line a, b, is to be regarded as a composite line, offering the same angle from each end, but different surge impedances.

Terminal Potentials and Currents in a Dissymmetrical System. If we have any physically consistent distributions of potential, current and power at the a and b terminals of a known dissymmetrical system, and we know the values of potential and current at one end, we can readily ascertain the corresponding values at the other, for one and the same frequency.

In the case of Fig. 15, we have the same dissymmetrical system as in Figs. 11, 12 and 14, for which θ, z_{0a} and z_{0b} are given. A potential of 1.0 volt and an entering current of $I_a = 0.347826$ milliampere, are found at the generating end a. Required the corresponding values at the motor end b. The value of the load σ at b need not be known. We have then:



FIGS. 15 AND 16—NETWORK LOADED WITH A TERMINAL IMPEDANCE σ AT EACH END ALTERNATELY

$$E_b = \frac{1}{q}\left(E_a \cosh \theta - I_a\, z_{0a} \sinh \theta\right) \text{ volts } \angle \quad (7)$$

$$\text{and} \quad I_b = q\left(I_a \cosh \theta - E_a\, y_{0a} \sinh \theta\right) \text{ amperes } \angle \quad (8)$$

where $y_{0a} = 1/z_{0a}$, is the surge admittance, as measured from a. If $q = 1$; so that the dissymmetry is made to disappear, the above formulas are those which apply to any ordinary smooth line of θ and z_0. In the case considered, $E_b = 0.21739$ volt, and $I_b = 0.21739$ milliampere.

If on the contrary, we know E_b and I_b only, without necessarily knowing the physical conditions at the motor terminal b, we may proceed to find the corresponding values at the generator end a. We obtain:

$$E_a = q\,(E_b \cosh\theta + I_b z_{0b} \sinh\theta) \qquad \text{volts } \angle \quad (9)$$

and

$$I_a = \frac{1}{q}\,(I_b \cosh\theta + E_b y_{0b} \sinh\theta) \qquad \text{amperes } \angle \quad (10)$$

where $y_{0b} = 1/z_{0b}$ is the surge admittance as measured at b. If we make $q = 1$, these formulas become identical with those pertaining to a simple smooth line and symmetrical system. In any such smooth-line case, formulas like (7), (8), (9) and (10) can be employed for any point along the line; but in the case of a dissymmetrical system, they apply only to the terminals a and b.

Again, if b is the generator end, and a the motor end of the dissymmetrical system, (see Fig. 16), and we know the potential E_a and current I_a, at a, we can find the corresponding values at b as follows:

$$E_b = \frac{1}{q}\,(E_a \cosh\theta + I_a z_{0a} \sinh\theta) \qquad \text{volts } \angle \quad (11)$$

$$I_b = q\,(I_a \cosh\theta + E_a y_{0a} \sinh\theta) \qquad \text{amperes } \angle \quad (12)$$

Similarly, if we know the generator-end values, E_b and I_b, the corresponding motor-end values E_a and I_a become:

$$E_a = q\,(E_b \cosh\theta - I_b z_{0b} \sinh\theta) \qquad \text{volts } \angle \quad (13)$$

$$I_a = \frac{1}{q}\,(I_b \cosh\theta - E_b y_{0b} \sinh\theta) \qquad \text{amperes } \angle \quad (14)$$

Here again, if $q = 1$, the formulas are those which apply at the terminals of a smooth line and symmetrical system. Hence we may formulate the following rule for any dissymmetrical case. Having given the potential and current at one end, write the corresponding values for the other, with regard to the direction of power transmission, as though the line were smooth and the system symmetrical. Use the surge impedance or admittance belonging to the end at which the values are known. Then an unknown potential E_a at a, will be $q = \sqrt{z_{0a}/z_{0b}}$ times that found by the regular formula, and likewise an unknown current I_b at b. On the other hand, an unknown current I_a at a, or a potential E_b at b, will be $1/q$ times that found by the regular smooth-line formula. The power products $E_a I_a$, and $E_b I_b$ will therefore not differ from those given by the corresponding smooth-line formulas.

Position Angles at the Terminals of a Dissymmetrical System. If we load the motor end b of a dissymmetrical system, like that of Fig. 15, with an impedance σ ohms $\angle$, we may find the position angle of that end in the same manner as though the load were applied to the motor end of a simple smooth line; *i. e.*

$$\tanh\delta_b = \sigma/z_{0b} \qquad \text{numeric } \angle \quad (15)$$

Thus in Fig. 15, the position angle δ_b at b is 0.228437 hyp. The position angle δ_a at the generator end a is then obtained by the usual rule:

$$\delta_a = \theta + \delta_b \qquad \text{hyps } \angle \quad (16)$$

which in this case is 0.980217 hyp. The potentials and currents at the ends of the dissymmetrical system are then related as follow:

With E_b and I_b known,

$$E_a = q\left(E_b \frac{\sinh\delta_a}{\sinh\delta_b}\right) \qquad \text{volts } \angle \quad (17)$$

$$I_a = \frac{1}{q}\left(I_b \frac{\cosh\delta_a}{\cosh\delta_b}\right) \qquad \text{amperes } \angle \quad (18)$$

The impedance beyond b is σ. The impedance at a is:

$$Z_a = z_{0a} \tanh\delta_a \qquad \text{ohms } \angle \quad (19)$$

The power at a is the local product of E_a and I_a in vector watts, according to the regular rule.

With E_a and I_a known,

$$E_b = \frac{1}{q}\left(E_a \frac{\sinh\delta_b}{\sinh\delta_a}\right) \qquad \text{volts } \angle \quad (20)$$

$$I_b = q\left(I_a \frac{\cosh\delta_b}{\cosh\delta_a}\right) \qquad \text{amperes } \angle \quad (21)$$

Formulas (17) to (21) apply to ordinary smooth-line symmetrical systems, if $q = 1$. Whereas, however, in a smooth-line system, a position angle can be assigned to each and every point of the line, we are only justified in assigning position angles to two points—the a and b ends—of a dissymmetrical system.

Similarly, if b is the generator end (Fig. 16) and a the motor end of a dissymmetrical system, to which a load σ is applied, the position angle at a is defined by:

$$\tanh\delta_a = \sigma/z_{0a} \qquad \text{numeric } \angle \quad (22)$$

In the case considered, $\delta_a = 0.268221$ hyp. The position angle at b is:

$$\delta_b = \theta + \delta_a \qquad \text{hyps } \angle \quad (23)$$

The potentials and currents at the two ends are thus related:

With E_b and I_b known,

$$E_a = q\left(E_b \frac{\sinh\delta_a}{\sinh\delta_b}\right) \qquad \text{volts } \angle \quad (24)$$

$$I_a = \frac{1}{q}\left(I_b \frac{\cosh\delta_a}{\cosh\delta_b}\right) \qquad \text{amperes } \angle \quad (25)$$

With E_a and I_a known,

$$E_b = \frac{1}{q}\left(E_a \frac{\sinh\delta_b}{\sinh\delta_a}\right) \qquad \text{volts } \angle \quad (26)$$

$$I_b = q\left(I_a \frac{\cosh\delta_b}{\cosh\delta_a}\right) \qquad \text{amperes } \angle \quad (27)$$

The impedance at the generator end is:

$$Z_b = z_{0b} \tanh\delta_b \qquad \text{ohms } \angle \quad (28)$$

Formulas (24) to (27) are identical with (17), (18) and (20), (21).

We may formulate the deductions from the last 13 formulas as follow: In any dissymmetrical system for which θ, z_{0a} and z_{0b} are given, and which is loaded at the motor end with an impedance σ, find the position angle at the motor end, with reference to that end's surge impedance. If the potential and current are given at one end of the system, the values at the other end are found by using the ordinary position-angle formulas for a smooth line. An unknown E_a, or I_b, will then be $q = \sqrt{z_{0a}/z_{0b}}$ times the value so obtained. On the other hand, an unknown E_b or I_a, will be $1/q$ times the value obtained by the ordinary formula. This rule agrees with that given above for use with formulas (**7**) to (**14**).

In view of the above mentioned reciprocal factors for potential and current at each terminal, leaving the power products unchanged, it becomes possible to reduce a dissymmetrical system to a corresponding symmetrical system plus a terminal ideal transformer, without losses, and with a transformation ratio of q. It is doubtful, however, there would be any advantage in this plan over the method above outlined.

In the case of an actual composite line, like that of Fig. 4, with definite sections, having respective angles $\theta_1, \theta_2, \theta_3 \ldots$ and surge impedances $z_{01}, z_{02}, z_{03}, \ldots$, it is advantageous to work up the position angles at the successive terminals commencing with the motor end. The terminal position angle at a would thus become δ_1, with a terminal surge impedance z_{01}; while working from the other end, the terminal position angle would be δ_n, with a terminal surge impedance of z_{0n}. The impedance R_{ag} as measured at a would be

$$R_{ag} = z_{01} \tanh \delta_1 \qquad \text{ohms} \angle \qquad (\mathbf{29})$$

while that measured at b would be

$$R_{bg} = z_{0n} \tanh \delta_n \qquad \text{ohms} \angle \qquad (\mathbf{30})$$

These values are not inconsistent with (**19**) and (**28**), which pertain to a general network; whereas (**29**) and (**30**) pertain to a particular composite line with detailed sections.

Receiving-End Impedance of a Dissymmetrical Network. If we define this quantity as the ratio of the e. m. f. impressed at the generator terminals to the current received through a load connecting the motor terminals, we find that for a load σ at b, (Fig. 15), the receiving end impedance is:

$$Z_{lb} = q\,(z_{0b} \sinh \theta + \sigma \cosh \theta) \qquad \text{ohms} \angle \qquad (\mathbf{31})$$

In the case represented in Fig. 15, $Z_{lb} = 4600.0$ ohms. When $q = 1$, this expression reduces to the ordinary value $z_0 \sinh \theta + \sigma \cosh \theta$, for a simple smooth line or symmetrical system.

When $\sigma = 0$, or the b terminals are short-circuited, (**31**) becomes:

$$Z_{lb} = q\, z_{0b} \sinh \theta = z_{ab} \sinh \theta \qquad \text{ohms} \angle \qquad (\mathbf{32})$$

which is the architrave impedance in Fig. 12, or 3400.0 ohms.

Similarly, for a load of σ ohms $\angle$ at the motor end a, the receiving-end impedance becomes (Fig. 16.)

$$Z_{la} = \frac{1}{q}\,(z_{0a} \sinh \theta + \sigma \cosh \theta) \qquad \text{ohms} \angle \qquad (\mathbf{33})$$

In the case of Fig. 16, this becomes 4800.0 ohms. When $q = 1$, the expression (**33**) again coincides with the ordinary smooth-line formula. Moreover, when $\sigma = 0$,

$$Z_{la} = \frac{1}{q}\,(z_{0a} \sinh \theta) = z_{ab} \sinh \theta \qquad \text{ohms} \angle \qquad (\mathbf{34})$$

or 3400.0 ohms, the architrave impedance of the equivalent dissymmetrical π.

We may formulate these results as follows. The receiving-end impedance of a dissymmetrical network loaded at the receiving end, is found by the usual rule for a terminally loaded smooth line, using the surge impedance of the motor end. The result must then be multiplied by q, when the generator is at a, and by $1/q$ when the generator is at b. In case the load is short-circuited, the receiving-end impedance reduces to $\rho'' = z_{ab} \sinh \theta$, the architrave impedance of the equivalent dissymmetrical π.

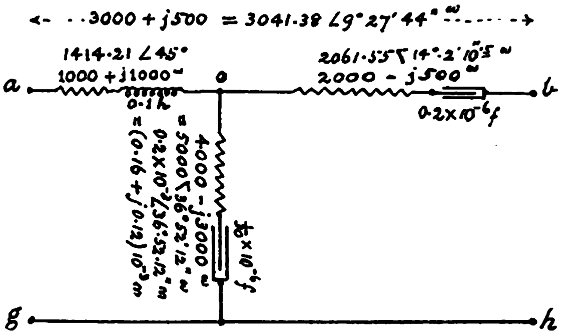


FIG. 17—DISSYMMETRICAL T OF IMPEDANCE REPRESENTING A NETWORK OF ALTERNATING-CURRENT CONDUCTORS

Technique of Measurements. We have assumed that the measurements made on the network are the four terminal impedances R_{af}, R_{ag}, R_{bf}, and R_{bg}. In cases where there is but little difference between R_{af} and R_{ag}, or between R_{bf} and R_{bg}, the value of θ may not admit of being determined with satisfactory precision; although the value of z_{ab} will usually be acceptable. In such a case, if the two ends of the network can be brought to the same testing table, it may be preferable to adopt the following technique:

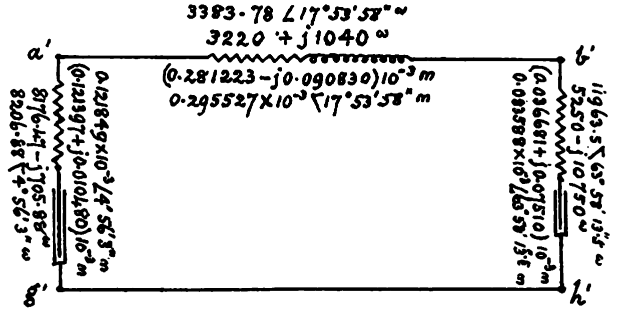


FIG. 18—DISSYMMETRICAL EQUIVALENT π OF SAME

1. Measure, with b open, the impedance R_{af} ohms at a, and the entering current I_a at a. At the same time measure the vector p. d. e_b volts thereby produced at the b terminals. From these data, ρ_1 and $\mathcal{R}$ can be deduced.

2. Measure R_{ag} ohms ∠ at a. With this value and R_{af}, find z_{0a}, by (2).
3. Repeat the measurements (1) and (2) reciprocally with the ends of the network reversed, so as to obtain R_{bf}, e_a and R_{bg}. These will give ρ_2 and z_{0b}, together with check values of θ and $\mathfrak{R}$.

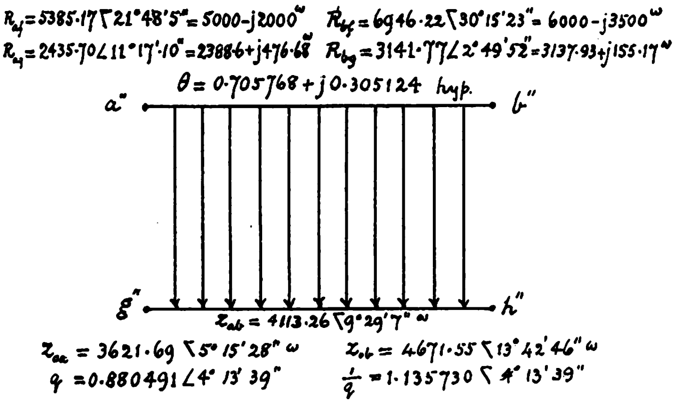

R_af = 5385.17 ⦣21°48′5″ = 5000 − j2000 ω
R_bf = 6946.22 ⦣30°15′23″ = 6000 − j3500 ω
R_ag = 2435.70 ∠11°17′.10″ = 2388.6 + j476.68 ω
R_bg = 3141.77 ∠2°49′52″ = 3137.93 + j155.17 ω
θ = 0.705768 + j0.305124 hyp.
a″
b″
g″
h″
z_ab = 4113.26 ⦣9°29′7″ ω
z_oa = 3621.69 ⦣5°15′28″ ω
z_ob = 4671.55 ⦣13°42′46″ ω
q = 0.880491 ∠4°13′39″
1/q = 1.135730 ⦣4°13′39″

FIG. 19—NETWORK OF FIG. 17 WITH SINGLE ANGLE θ, BUT WITH SURGE IMPEDANCES z_{0a} AND z_{0b} AS DETERMINED FROM EACH END.

Example of a Dissymmetrical Alternating-Current System. Fig. 17 offers a particular dissymmetrical T system, which might be the equivalent of a certain network between the two pairs of terminals a, g and b, h. The impedance $a\,o$, is $1000 + j\,1000$ ohms, such as might be produced by an inductor of 0.1 henry and 1000 ohms, at an angular velocity of $\omega = 10000$ radians per second, (1591.5 ∼). The impedance $o\,b$ is $2000 - j\,500$ ohms, such as might be produced at the same frequency by a condenser of 0.2 microfarad, in series with 2000 ohms. The impedance $o\,g$ is $4000 - j\,3000$ ohms. The equivalent π of this system is given in Fig. 18,,and the equivalent composite line network in Fig. 19. The angle subtended by the network is $\theta = 0.705768 + j\,0.305124$ hyps; or quadranting the imaginary part $\theta = 0.705768 + j\,\underline{0.194248}$ hyps. The functions of this angle are:

$$\sinh\theta = 0.822652 \angle 27^\circ\, 23'\, 05''$$
$$\cosh\theta = 1.223218 \angle 10^\circ\, 50'\, 28''$$
$$\tanh\theta = 0.672531 \angle 16^\circ\, 32'\, 37''$$

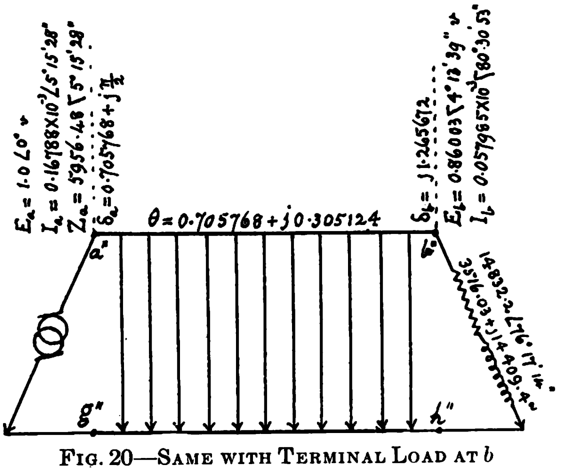
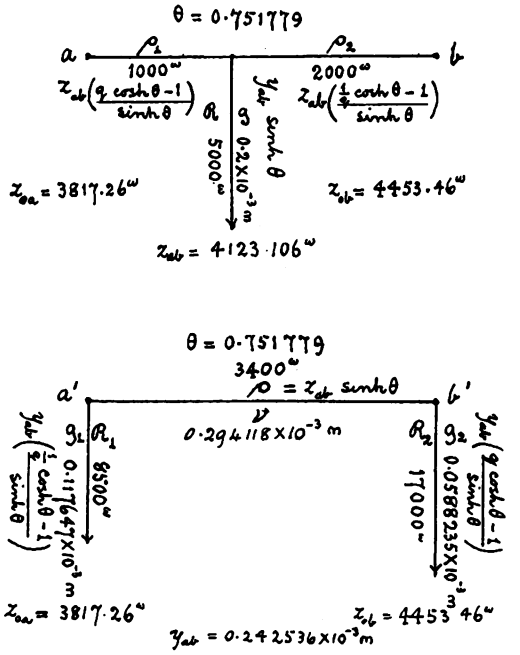


FIG. 20—SAME WITH TERMINAL LOAD AT b

Many of the formulas in Appendix A can be checked from Figs. (17), (18) and (19) by an inspection of the slopes of the various complex quantities presented.

In Fig. 20, the system of Figs. (17), (18) and (19) is represented as being loaded at the b end with $\sigma = 3516.03 + j\,14409.4$ ohms, and supplied at the a end with an e. m. f. of 1.0 ∠ 0° volt at the frequency of reference. The terminal position angles, potentials and currents are marked on the Figure, in accordance with formulas (15) to (21).

In Appendix A, a number of formulas have been collected relating to dissymmetrical networks and their equivalent circuits. The notation is in accordance with Figs. 21 and 22, and the numerical values pertain to the particular case of Figs. 14, 15 and 16.

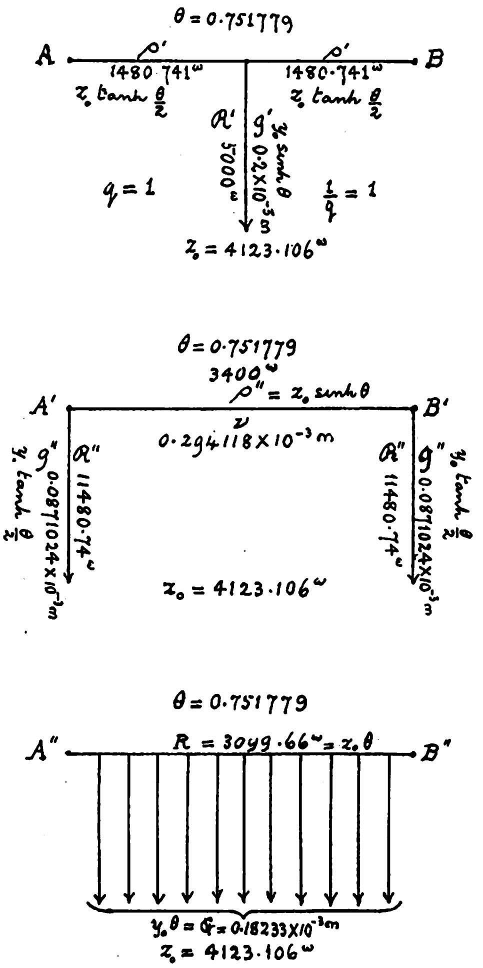


FIGS. 21 AND 22—DISSYMMETRICAL T AND π WITH THEIR ELEMENTS SYMBOLIZED, ALSO NUMERICAL VALUES CONFORMING WITH THE PARTICULAR CASE OF FIGS. 11, 12 AND 14.

In Appendix B, a corresponding series of formulas has been collected, relating to symmetrical networks and their equivalent circuits; *i. e.*, for the case $q = 1$. The notation is in accordance with Figs. 23, 24 and 25. The numerical values pertain to the system having the same θ as in Figs. 14, 15 and 16 and with its surge impedance z_0 equal to the geomean surge impedance z_{ab} of the dissymmetrical system. The values of R_f and R_g in this case are respectively the geomean values of R_{af}, R_{bf} and R_{ag}, R_{bg}.

We have considered, in all of the cases represented by the figures, that in any dissymmetrical T, all of the series impedance is lodged in the conductor a, o, b, and that the return conductor g, h, has negligible impedance. According to the known theory of T conductors, however, any desired share of series impedance can be transferred from a, o, b, to g, h, without disturbing the e. m. fs., currents and powers at and beyond the system terminals, provided that the total series impedance on the a side of the leak $o\ q$ is constant, and likewise the total series impedance on the b side. In other words a T-system may be converted into a H-system.

Similarly, although in all the π-systems indicated in the Figures, all the series impedance is lodged in the conductor a', b'; yet any desired share of this impedance may be transferred to g', h', without disturbing the e. m. fs., currents and powers at and beyond the system terminals. In other words, the dissymmetrical π s may be converted into corresponding O-systems.



FIGS. 23, 24 AND 25—SYMMETRICAL T, π AND SMOOTH LINE HAVING THE SAME ANGLE FEEDER AS IN FIGS. 14, 21 AND 22, ALSO WITH $z_o = z_{ab}$ OF THE DISSYMMETRICAL CASE.

SUMMARY OF CONCLUSIONS

1. Any dissymmetrical conducting network, each element of which obeys Ohm's law generalized for alternating currents, may be reduced to symmetry, with respect to two pairs of terminals, and at an assigned frequency, either by finding its equivalent T and equating the two arms, through the medium of a permanent series terminal load; or, by finding its equivalent π and equating the two pillars, through the medium of a permanent terminal leak load. The system then offers a certain angle θ, a surge impedance z_0, and a single terminal load, which may be treated as having an auxiliary angle θ'.

2. Any such dissymmetrical network may also be regarded as possessing and being defined by (a) an angle θ, (b) a geomean surge impedance z_{ab}, and (c) an inequality ratio q. The electric behavior of the network, at the assigned frequency, with respect to the two pairs of terminals, may then be computed by the usual formulas applying to a simple smooth line; but with the further application of the coefficient q.

3. The theory of simple alternating-current lines with smoothly distributed constants may thus be regarded as being a particular case ($q = 1$) of the general theory of dissymmetrical networks having any degree of complexity.

LIST OF SYMBOLS EMPLOYED

δ_a, δ_b,	position angles at a and b ends of a diss. network (hyps $\angle$).
δ_1, δ_n	position angles at ends of a detailed composite line (hyps $\angle$).
E_a, E_b	e. m. fs. applied to a and b ends of a diss. network (volts $\angle$).
e_a, e_b,	p. d. s at open a and b ends of diss. network by application of a testing e. m. f. at the opposite end (volts $\angle$).
f	frequency applied to network (cycles per second).
G	total conductance of a smooth sym. line (mhos $\angle$).
G_a, G_b,	admittance of a sym. system, when respectively freed and grounded at the other end (mhos $\angle$).
G_{af}, G_{ag},	admittance of a diss. system, at the a end, when respectively freed and grounded at the other end (mhos $\angle$).
G_{bf}, G_{bg},	admittance of a diss. system, at the b end, when respectively freed and grounded at the other end (mhos $\angle$).
g	admittance of leak in equiv. T of a diss. system (mhos $\angle$).
g_1, g_2,	admittances of leaks at a and b ends in equiv. π of a diss. system (mhos $\angle$).
g'	admittance of leak in equiv. T of a sym. system (mhos $\angle$).
g''	admittance of each leak in equiv. π of a sym. system (mhos $\angle$).
θ	angle subtended by a line or system (hyps $\angle$).
θ'	angle subtended by a terminal load (hyps $\angle$).
$\theta_1, \theta_2, \ldots \theta_n$	angles subtended by successive sections of a composite line (hyps $\angle$).
I_a, I_b,	current strengths at ends of a network (amperes $\angle$)
$j = \sqrt{-1}$	
ν	series adm. of a sym. or diss. π (mhos $\angle$).
$q = \sqrt{z_{0a}/z_{0b}}$	inequality ratio of a diss. network (numeric $\angle$).

R	total conductor resistance of a smooth line (ohms ∠).
R_f, R_g,	impedance at one end of a sym. system, when the other end is freed and grounded respectively (ohms ∠).
R_{af}, R_{ag},	impedance at the *a* end of a diss. system when the other end is freed and grounded respectively (ohms ∠).
R_{bf}, R_{bg},	impedance at the *b* end of a diss. system, when the other end is respectively freed and grounded (ohms ∠).
$\mathcal{R}$	impedance of leak in equiv. *T* of a diss. system (ohms ∠).
$\mathcal{R}_1$, $\mathcal{R}_2$,	impedances of *a* and *b* leaks in the equiv. π of a diss. system (ohms ∠).
$\mathcal{R}'$	impedance of leak in equiv. *T* of a sym. system (ohms ∠).
$\mathcal{R}''$	impedance of each leak in equiv. π of a sym. system (ohms ∠).
ρ	impedance of architrave in equiv. π of a diss. system (ohms ∠).
ρ_1, ρ_2,	impedance of *a* and *b* arms in equiv. *T* of a diss. system (ohms ∠).
ρ'	impedance of each arm in equiv. *T* of a sym. system (ohms ∠).
ρ''	impedance of architrave in equiv. π of a sym. system (ohms ∠).
σ	impedance of a load at motor end of a system (ohms ∠).
y_0	surge admittance of a sym. system (mhos ∠).
y_{0a}, y_{0b},	surge admittance at *a* and *b* ends of a diss. system (mhos ∠).
$y_{ab} = \sqrt{y_{0a} y_{0b}}$	geometrical mean surge admittance of a diss. system mhos ∠).
Z_a, Z_b,	generator-end impedances of a diss. system (ohms ∠).
Z_{la}, Z_{lb},	receiving-end impedances of a diss. system, including load (ohms ∠).
z_0,	surge impedance of a sym. system (ohms ∠).
z_{0a}, z_{0b},	surge impedance at *a* and *b* ends of a diss. system (ohms ∠).
$z_{ab} = \sqrt{z_{0a} z_{0b}}$	geomean surge impedance of a diss. system (ohms ∠).
$z_{01}, z_{02}, \ldots z_{0n}$	surge impedances of successive sections in composite line (ohms ∠).
$\omega = 2\pi f$	angular velocity of frequency impressed on a system (rad. per sec.)
~	sign for "cycles per second."
∠	sign for a complex numerical quantity.
hyp.	abbreviation for "hyperbolic radian."

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Without pretensions as to completeness

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APPENDIX A

List of formulas applicable to a dissymmetrical *T*, π or Network. The numerical values pertain to the case of Figs. 17, 18, 21 and 22.

$$R_{ag}/R_{af} = R_{bg}/R_{bf} = G_{af}/G_{ag} = G_{bf}/G_{bg} = \frac{\mathcal{R}(\rho_1+\rho_2)+\rho_1\rho_2}{(\mathcal{R}+\rho_1)(\mathcal{R}+\rho_2)} = \frac{\nu(g_1+g_2)+g_1 g_2}{(\nu+g_1)(\nu+g_2)} = \tanh^2\theta = 0.404762 \quad \text{numeric } \angle \quad (35)$$

$$\rho_1/\rho_2 = g_2/g_1 = \mathcal{R}_1/\mathcal{R}_2 = \frac{q\cosh\theta-1}{\frac{1}{q}\cosh\theta-1} = 0.5 \quad \text{numeric } \angle \quad (36)$$

$$q = \sqrt{R_{ag}/R_{bg}} = \sqrt{R_{af}/R_{bf}} = \sqrt{G_{bg}/G_{ag}} = \sqrt{G_{bf}/G_{af}} = \sqrt{z_{0a}/z_{0b}} = \sqrt{\frac{\rho_1+\mathcal{R}}{\rho_2+\mathcal{R}}} = \sqrt{\frac{g_2+\nu}{g_1+\nu}} = z_{0a}/z_{ab} = 0.92582 \quad \text{numeric } \angle \quad (37)$$

$$1/q = \sqrt{R_{bg}/R_{ag}} = \sqrt{R_{bf}/R_{af}} = \sqrt{G_{ag}/G_{bg}} = \sqrt{G_{af}/G_{bf}} = \sqrt{z_{0b}/z_{0a}} = \sqrt{\frac{\rho_2+\mathcal{R}}{\rho_1+\mathcal{R}}} = \sqrt{\frac{g_1+\nu}{g_2+\nu}} = z_{0b}/z_{ab} = 1.08012 \quad \text{numeric } \angle \quad (38)$$

$$\frac{\frac{1}{2}(\rho_1+\rho_2)}{\mathcal{R}} = \frac{\frac{1}{2}(g_1+g_2)}{\nu} = \left(\frac{q+1/q}{2}\right)\cosh\theta - 1 = 0.3 \quad \text{numeric } \angle \quad (39)$$

$$\frac{\sqrt{\rho_1\rho_2}}{\mathcal{R}} = \frac{\sqrt{g_1 g_2}}{\nu} = \sqrt{(q\cosh\theta-1)\left(\frac{1}{q}\cosh\theta-1\right)} = 0.28284 \quad \text{numeric } \angle \quad (40)$$

$$\begin{aligned}\rho_1/g_2 \quad &= \rho_2/g_1 = \rho_1\,\mathcal{R}_2 = \rho_2\,\mathcal{R}_1 = \rho\,\mathcal{R} = \rho/g = \mathcal{R}/\nu = \sqrt{\frac{\rho_1\,\rho_2}{g_1\,g_2}} = z_{0a}\,z_{0b} = z_{ab}^2 = R_{af}\,R_{bg}\\ &= R_{ag}\,R_{bf} = \rho_1\,\rho_2 + \mathcal{R}\,(\rho_1+\rho_2) = \sqrt{R_{af}\,R_{ag}\,R_{bf}\,R_{bg}} = \sqrt{\rho_1\,\rho_2\,\mathcal{R}_1\,\mathcal{R}_2}\\ &= \sqrt{\{\,\rho_1\,(\rho_2+\mathcal{R}) + \rho_2\,\mathcal{R}\,\}\ \{\,\rho_2\,(\rho_2+\mathcal{R}) + \rho_1\,\mathcal{R}\,\}} = 17{,}000{,}000 = (4123.106)^2\\ &\qquad\qquad \text{ohms}^2\angle \quad (41)\end{aligned}$$

$$\begin{aligned}g_1/\rho_2 \quad &= g_2/\rho_1 = \nu\,g = g/\rho = \nu/\mathcal{R} = \sqrt{\frac{g_1\,g_2}{\rho_1\,\rho_2}} = y_{0a}\,y_{0b} = y_{ab}^2 = G_{af}\,G_{bg} = G_{ag}\,G_{bf}\\ &= g_1\,g_2 + \nu\,(g_1+g_2) = \sqrt{G_{af}\,G_{ag}\,G_{bf}\,G_{bg}} = \sqrt{\{g_1\,(g_2+\nu) + g_2\,\nu\}\ \{g_2\,(g_1+\nu) + g_1\,\nu\}}\\ &= 0.058824\times10^{-6} = (0.2425355\times10^{-3})^2 \qquad \text{mhos}^2\angle \quad (42)\end{aligned}$$

$$z_{0a} \quad = \sqrt{R_{af}\,R_{ag}} = \sqrt{\rho_1\,(\rho_1+\mathcal{R}) + \rho_2\,\mathcal{R}\,\frac{\rho_1+\mathcal{R}}{\rho_2+\mathcal{R}}} = q\,z_{ab} = q^2\,z_{0b} = 3817.26 \qquad \text{ohms}\ \angle \quad (43)$$

$$z_{0b} \quad = \sqrt{R_{bf}\,R_{bg}} = \sqrt{\rho_2\,(\rho_2+\mathcal{R}) + \rho_1\,\mathcal{R}\,\frac{\rho_2+\mathcal{R}}{\rho_1+\mathcal{R}}} = \frac{1}{q}z_{ab} = \frac{1}{q^2}\,z_{0a} = 4453.46 \qquad \text{ohms}\ \angle \quad (44)$$

$$\begin{aligned}\sinh\theta \quad &= \sqrt{\frac{R_{ag}}{R_{af}-R_{ag}}} = \sqrt{\frac{R_{bg}}{R_{bf}-R_{bg}}} = \frac{\sqrt{\mathcal{R}\,(\rho_1+\rho_2) + \rho_1\,\rho_2}}{\mathcal{R}} = \frac{\sqrt{\nu\,(g_1+g_2) + g_1\,g_2}}{\nu}\\ &= \sqrt{\rho/\mathcal{R}} = \sqrt{\rho\,g} = z_{ab}/\mathcal{R} = y_{ab}/\nu = 0.82462 \qquad \text{numeric}\ \angle \quad (45)\end{aligned}$$

$$\begin{aligned}\cosh\theta \quad &= \sqrt{\frac{R_{af}}{R_{af}-R_{ag}}} = \sqrt{\frac{R_{bf}}{R_{bf}-R_{bg}}} = \frac{\sqrt{(\mathcal{R}+\rho_1)(\mathcal{R}+\rho_2)}}{\mathcal{R}} = \sqrt{1+\rho\,g}\\ &= \frac{\sqrt{(\nu+g_1)(\nu+g_2)}}{\nu} = \frac{\sqrt{R_{af}\,R_{bf}}}{\mathcal{R}} = \frac{\sqrt{G_{ag}\,G_{bg}}}{\nu} = 1.296148 \qquad \text{numeric}\ \angle \quad (46)\end{aligned}$$

$$\begin{aligned}\tanh\theta \quad &= \sqrt{R_{ag}/R_{af}} = \sqrt{R_{bg}/R_{bf}} = = \sqrt{\frac{\mathcal{R}\,(\rho_1+\rho_2) + \rho_1\,\rho_2}{\mathcal{R}^2 + \mathcal{R}\,(\rho_1+\rho_2) + \rho_1\,\rho_2}}\\ &= \sqrt{\frac{\nu\,(g_1+g_2) + g_1\,g_2}{\nu^2 + \nu\,(g_1+g_2) + g_1\,g_2}} = \frac{z_{ab}}{\sqrt{R_{af}\,R_{bf}}} = \frac{y_{ab}}{\sqrt{G_{ag}\,G_{bg}}} = 0.636209 \qquad \text{numeric}\ \angle \quad (47)\end{aligned}$$

$$\rho_1/\mathcal{R} \quad = \rho_1\,g = g_2/\nu = g_2\,\rho = \rho/\mathcal{R}_2 = q\cosh\theta - 1 = 0.2 \qquad \text{numeric}\ \angle \quad (48)$$

$$\rho_2/\mathcal{R} \quad = \rho_2\,g = g_1/\nu = g_1\,\rho = \rho/\mathcal{R}_1 = \frac{1}{q}\cosh\theta - 1 = 0.4 \qquad \text{numeric}\ \angle \quad (49)$$

$$\rho_1 \quad = z_{ab}\left(\frac{q\cosh\theta - 1}{\sinh\theta}\right) = 1000 \qquad \text{ohms}\ \angle \quad (50)$$

$$\rho_2 \quad = z_{ab}\left(\frac{\frac{1}{q}\cosh\theta - 1}{\sinh\theta}\right) = 2000 \qquad \text{ohms}\ \angle \quad (51)$$

$$\mathcal{R} \quad = z_{ab}\operatorname{cosech}\theta = \sqrt{R_{bf}\,(R_{af}-R_{ag})} = \sqrt{R_{af}\,(R_{bf}-R_{bg})} = 5000 \qquad \text{ohms}\ \angle \quad (52)$$

$$g \quad = y_{ab}\sinh\theta = 0.2\times10^{-3} \qquad \text{mhos}\ \angle \quad (53)$$

$$\begin{aligned}\rho \quad &= z_{ab}\sinh\theta = \sqrt{\frac{R_{bg}}{G_{ag}-G_{af}}} = \sqrt{\frac{R_{ag}}{G_{bg}-G_{bf}}} = R_{ag}\sqrt{\frac{R_{bf}}{R_{af}-R_{ag}}} = R_{bg}\sqrt{\frac{R_{af}}{R_{bf}-R_{bg}}}\\ &= \frac{R_{af}\,R_{bg}}{\mathcal{R}} = \frac{R_{ag}\,R_{bf}}{\mathcal{R}} = 3400 \qquad \text{ohms}\ \angle \quad (54)\end{aligned}$$

$$\begin{aligned}\nu \quad &= y_{ab}\operatorname{cosech}\theta = \sqrt{G_{bg}\,(G_{ag}-G_{af})} = \sqrt{G_{ag}\,(G_{bg}-G_{bf})} = \frac{G_{af}\,G_{bg}}{g} = \frac{G_{ag}\,G_{bf}}{g}\\ &= 0.294118\times10^{-3} \qquad \text{mhos}\ \angle \quad (55)\end{aligned}$$

$$g_1 \quad = y_{ab}\left(\frac{\frac{1}{q}\cosh\theta - 1}{\sinh\theta}\right) = 0.117647\times10^{-3} \qquad \text{mhos}\ \angle \quad (56)$$

$$g_2 = y_{ab}\left(\frac{q\cosh\theta - 1}{\sinh\theta}\right) = 0.058824 \times 10^{-3} \qquad \text{mhos} \angle \quad (57)$$

$$R_{ag} = z_{0a}\tanh\theta = q\, z_{ab}\tanh\theta = q^2 z_{0b}\tanh\theta = 2428.57 \qquad \text{ohms} \angle \quad (58)$$

$$R_{bg} = z_{0b}\tanh\theta = \frac{1}{q} z_{ab}\tanh\theta = \frac{1}{q^2} z_{0b}\tanh\theta = 2833.3 \qquad \text{ohms} \angle \quad (59)$$

$$R_{af} = z_{0a}\coth\theta = q\, z_{ab}\coth\theta = q^2 z_{0b}\coth\theta = 6000 \qquad \text{ohms} \angle \quad (60)$$

$$R_{bf} = z_{0b}\coth\theta = \frac{1}{q} z_{ab}\coth\theta = \frac{1}{q^2} z_{0b}\coth\theta = 7000 \qquad \text{ohms} \angle \quad (61)$$

APPENDIX B

List of formulas applicable to a symmetrical T, π, or network, corresponding to the formulas of Appendix A. The numerical values pertain to the case of Figs. 23, 24 and 25.

$$R_g/R_f = G_f/G_g = \frac{\rho'(\rho' + 2\mathcal{R}')}{(\rho' + \mathcal{R}')^2} = \frac{g''(g'' + 2\nu)}{(g'' + \nu)^2} = \tanh^2\theta = 0.404762 \qquad \text{numeric} \angle \quad (35a)$$

$$\rho'/\mathcal{R}' = g''/\nu = \rho' g' = \rho'' g'' = \cosh\theta - 1 = \operatorname{versh}\theta = 2\sinh^2(\theta/2) = 0.296148 \quad \text{numeric} \angle \quad (39a)$$

$$\rho''/g' = \rho''\mathcal{R}' = \mathcal{R}'/\nu = \rho'(\rho' + 2\mathcal{R}') = R_f R_g = z_0^2 = (4123.106)^2 = 17{,}000{,}000 \quad \text{ohms}^2 \angle \quad (41a)$$

$$g'/\rho'' = \nu g' = \nu/\mathcal{R}' = g''(g'' + 2\nu) = G_f G_g = y_0^2 = (0.2425355 \times 10^{-3})^2 = 0.058824 \times 10^{-6} \qquad \text{mho}^2 \angle \quad (42a)$$

$$\sinh\theta = \sqrt{\frac{R_g}{R_f - R_g}} = \frac{\sqrt{\rho'(\rho' + 2\mathcal{R}')}}{\mathcal{R}'} = \frac{\sqrt{g''(g'' + 2\nu)}}{\nu} = z_0/\mathcal{R}' = y_0/\nu = \sqrt{g'\rho''} = \sqrt{\rho''/\mathcal{R}'} = 0.82462 \qquad \text{numeric} \angle \quad (45a)$$

$$\cosh\theta = \sqrt{\frac{R_f}{R_f - R_g}} = \frac{\rho' + \mathcal{R}'}{\mathcal{R}'} = \frac{g'' + \nu}{\nu} = R_f/\mathcal{R}' = G_g/\nu = \sqrt{1 + g'\rho''} = 1.296148 \quad \text{numeric} \angle \quad (46a)$$

$$\tanh\theta = \sqrt{R_g/R_f} = \frac{\sqrt{\rho'(\rho' + 2\mathcal{R}')}}{\rho' + \mathcal{R}'} = \frac{\sqrt{g''(g'' + 2\nu)}}{g'' + \nu} = z_0/R_f = y_0/G_g = 0.636209 \qquad \text{numeric} \angle \quad (47a)$$

$$\rho' = z_0\left(\frac{\cosh\theta - 1}{\sinh\theta}\right) = z_0\tanh(\theta/2) = 1480.741 \qquad \text{ohms} \angle \quad (50a)$$

$$\mathcal{R}' = z_0\operatorname{cosech}\theta = R_f\operatorname{sech}\theta = \sqrt{R_f(R_f - R_g)} = \frac{(\mathcal{R}'')^2}{\rho'' + 2\mathcal{R}''} = z_0^2/\rho'' = \frac{R_f R_g}{\rho''} = 5000 \qquad \text{ohms} \angle \quad (52a)$$

$$g' = y_0\sinh\theta = G_f\cosh\theta = \sqrt{\frac{G_f}{R_f - R_g}} = \frac{g''(g'' + 2\nu)}{\nu} = y_0^2/\nu = \frac{G_f G_g}{\nu} = 0.2 \times 10^{-3} \qquad \text{mhos} \angle \quad (53a)$$

$$\rho'' = z_0\sinh\theta = R_g\cosh\theta = R_g\sqrt{\frac{R_f}{R_f - R_g}} = \frac{\rho'(\rho' + 2\mathcal{R}')}{\mathcal{R}'} = z_0^2/\mathcal{R}' = 3400 \qquad \text{ohms} \angle \quad (54a)$$

$$\nu = y_0\operatorname{cosech}\theta = G_g\operatorname{sech}\theta = \sqrt{G_g(G_g - G_f)} = \frac{g''(g'' + 2\nu)}{g'} = y_0^2/g = \frac{G_f G_g}{g'} = 0.294118 \times 10^{-3} \qquad \text{mhos} \angle \quad (55a)$$

$$g'' = y_0\left(\frac{\cosh\theta - 1}{\sinh\theta}\right) = y_0\tanh(\theta/2) = 0.0871024 \times 10^{-3} \qquad \text{ohms} \angle \quad (56a)$$

$$R_g = z_0\tanh\theta = \frac{\rho'(\rho' + 2\mathcal{R}')}{\rho' + \mathcal{R}'} = 2623.157 \qquad \text{ohms} \angle \quad (58a)$$

$$R_f = z_0\coth\theta = \rho' + \mathcal{R}' = 6480.741 \qquad \text{ohms} \angle \quad (60a)$$

$$G_g = y_0\coth\theta = g'' + \nu = 0.38122 \times 10^{-3} \qquad \text{mhos} \angle \quad (62)$$

$$G_f = y_0\tanh\theta = \frac{g''(g'' + 2\nu)}{g'' + \nu} = 0.154149 \times 10^{-3} \qquad \text{mhos} \angle \quad (63)$$

LIST OF RESEARCH BULLETINS—Continued

Bulletin Number

17. Electromagnetic Theory of the Telephone Receiver, with Special Reference to Motional Impedance. A. E. Kennelly and H. Nukiyama; March, 1919.
18. A Rectangular-Component Two-Dimensional Alternating-Current Potentiometer. A. E. Kennelly and Edy Velander; July, 1919.
19. Current Distribution in Armature Conductors. Waldo V. Lyon; July, 1919.
20. Alternating-Current Planevector Potentiometer Measurements at Telephonic Frequencies. A. E. Kennelly and Edy Velander; July, 1919.
21. A Theory of Metallic Arc Welding. Ralph G. Hudson; March, 1920.
22. Oscillographs and Their Tests. A. E. Kennelly, R. N. Hunter and A. A. Prior; April, 1920.
23. A Direct Recording Method of Measuring Magnetic Flux Distribution. F. S. Dellenbaugh, Jr.; October, 1920.
24. The Transient Process of Establishing a Steadily Alternating Current on a Long Line, from Laboratory Measurements on an Artificial Line. A. E. Kennelly and U. Nabeshima; January, 1921.
25. A Dynamometrical Comparator. A Differential Dynamometer for Accurate Comparison of Alternating and Direct-Current Strengths. Edy Velander, February, 1921.
26. Induction Motor Core Losses. P. L. Alger and R. Eksergian, February, 1921.
27. Acoustic Impedance and Its Measurement. A. E. Kennelly and K. Kurokawa, February, 1921.
28. A Simple Harmonic Analyser. V. Bush; Electromechanical Device for Rapid Schedule Harmonic Analysis of Complex Waves. Frederick S. Dellenbaugh, Jr.; Alignment Chart for Circular and Hyperbolic Functions of a Complex Argument in Rectangular Coördinates. V. Bush, January, 1922.
29. A Mechanical Frequency-Meter of Telephonic Range. A. E. Kennelly and C. H. Manneback, January, 1922.
30. An Integral Equation for Skin Effect in Parallel Conductors. Charles Manneback, May, 1922.
31. The Effect of Absorbed Gas on the Conductivity of Glass. V. Bush and L. H. Connell, September, 1922.
32. On the Education of Engineers in France. A. E. Kennelly, January, 1923.
33. Heat Losses in Conductors. W. V. Lyon, March, 1923.
34. Another Harmonic Analyser. F. S. Dellenbaugh, Jr., March, 1923.
35. Radiation from Transmission Lines. C. H. Manneback, March, 1923.
36. Dissymetrical Electrical Networks. A. E. Kennelly, March, 1923.

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TRANSIENT CONDITIONS IN ELECTRIC MACHINERY

by
W. V. Lyon

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Transient Conditions in Electric Machinery

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Review of the Subject: *— The vector method is just as useful in solving problems involving transient conditions in electric circuits as it has proved to be when the currents and potentials are steady sinusoids. As far as the writer is aware, the vector method for determining transients in rotating electric machines was first used by L. Dreyfus. Previously the method had been applied to fixed combinations of resistances, inductances and capacitances by Kennelly and others.*

By making certain assumptions that are, however, quite reasonable in many cases, the transient currents in nearly all of the common types of electric machinery are damped sinusoids. Fortunately the damping is exponential and is thus readily accounted for. It is interesting to trace the development of the method. In the solution of all problems in direct currents the potentials, currents, and circuit constants are real numbers. In the corresponding problem in which the applied potentials are steady sinusoids, these quantities are all represented by complex numbers. In all other respects the working out of the solution is identical with that followed in the direct-current case. When the currents are damped sinusoids, they and the potentials and the circuit constants can still be represented by complex numbers. There is this difference, however; the vectors which represent the currents and potentials shrink exponentially as they rotate and the values of the circuit constants depend not only upon the frequency of the current, but also upon its rate of shrinking. Again the solution of any problem follows the same procedure that would the corresponding one in which the currents are steady sinusoids. In both the steady and damped sinusoidal cases the circuit constants depend upon the angular velocity of the vectors which represent the currents. In the former, the angular velocity is purely imaginary while in the latter it is complex, the real part being the rate at which the current vector shrinks and the imaginary portion, its angular velocity. In electric machinery in which rotating magnetic fields are produced, these fields shrink exponentially as they rotate when the currents are damped sinusoids. If these rotating magnetic fields are represented by vectors, the vectors will have a complex angular velocity just as do the currents. The e. m. f. which is produced by a steady sinusoidal variation of flux lags the flux by 90 degrees, whereas if the flux variation is a damped sinusoid, the angle of lag is less than 90 degrees, depending upon the damping. The mathematical relation, however, is the same, viz., the e. m. f. is proportional to the negative of the product of the flux and its angular velocity. It is then readily appreciated that the form of the solution for the transient state is the same as that which is used for the steady state. Before the method can be expected to give as accurate results as are obtained when predicting the steady operation, considerable experimental data must be obtained in order to determine the best methods of measuring the necessary constants, for these may be somewhat different during the transient period than during steady operation.

* * * * * * * *

THE problem of calculating the transient currents in alternating-current apparatus has rightly been considered a difficult one. Fortunately, the difficulty lies not so much in the problem itself as in the inadequacy of the mathematical treatment that has been employed. If a vector method of analysis is used, very great difficulties are removed. This was first appreciated by the author early in 1921 while studying the paper "Short-circuit Current of Induction Motors and Generators" by Doherty and Williamson. Later it was found that the vector method had already been developed by L. Dreyfus.[1] Notwithstanding this previous publication, the subject was considered of sufficient importance to warrant the presentation of this paper.

As is the case with many other complex problems, the method of attack is presented most clearly when certain ideal conditions are assumed. These conditions are that the resistances and inductances are constant and, in the case of rotating machines, that the speed is constant and that the space distribution of air-gap flux is sinusoidal. After one is familiar with the method in its simplest form, it may be possible to introduce refinements which will account for iron losses, variable permeability and harmonics.

The form of the transient current in any system of mutually inductive circuits is determined by the differential equations of the fall in electric pressure through them. With the foregoing assumptions in regard to the ideal character of the electric circuits, the differential equations which apply to most of the common types of direct and alternating-current machinery may be reduced to the form which is described as "linear with constant coefficients." This being true, the transient currents consist of one or more terms of the general form:

$$i = I \epsilon^{-\alpha t} \sin (\omega t + \theta)$$

The way in which this current varies depends upon the values of α and ω. There are four types of variations, only two of which occur as transients. If α and ω are both zero, it is a steady direct current of value, $I \sin \theta$. If α is a positive quantity, as it always is, and ω is zero, it is a diminishing direct current of value, $I \epsilon^{-\alpha t} \sin \theta$. If α is zero and ω is a positive quantity, it is a steady sinusoidal current of value, $I \sin (\omega t + \theta)$. And finally, if neither α nor ω are zero, it is a diminishing sinusoidal current. In each of these cases the current, whether steady or transient, may be represented by a vector which in general has a length of $I \epsilon^{-\alpha t}$ and which makes an angle of $(\omega t + \theta)$ with the horizontal axis. The projection of this vector on the vertical axis is the instantaneous value of the current. The four cases are illustrated in Fig. 1, *a*, *b*, *c* and *d*.

In each of the diagrams *I* marks the initial position of the vector. In the first case (*a*) the vector is fixed both in magnitude and in angular position. In the second case (*b*) the magnitude of the vector diminishes exponentially although its angular position does not change. In the third case (*c*) the vector has a constant

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[1] See Bibliography.

Presented at the Midwinter Convention of the A. I. E. E., New York, N. Y., February 14-17, 1923.

magnitude but rotates with an angular velocity of ω. In the fourth and general case (d) the magnitude of the vector diminishes exponentially while it rotates at a constant angular velocity of ω. Thus the general form of transient current may be represented by a vector which rotates at a constant angular velocity and whose terminus travels along a logarithmic spiral.

The velocity of the terminus of the vector has two components, one along the vector and one perpendicular to it. It is convenient to measure these component velocities in multiples of the instantaneous length of the vector. The term angular velocity is thus appropriate. This is a broader meaning of angular velocity than is ordinarily assigned to it, and for this reason, it is spoken of as the generalized angular velocity. Differentiation shows that the generalized angular velocity of this vector is, $m = (-\alpha + j\,\omega)$. See Fig. 2. Notice that the usual significance is attached to the symbol j. It indicates that the vector $j\,\omega$, lags the vector, $-\alpha$, by 90 degrees.

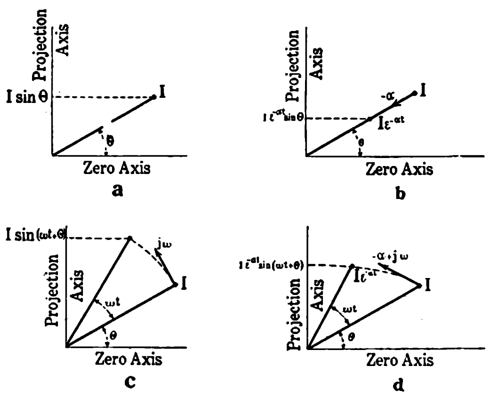

FIG. 1—THE END OF EACH VECTOR WHICH IS DRAWN FROM THE ORIGIN IS INDICATED BY A LARGE PERIOD; THE END OF SMALL VECTORS DRAWN FROM THESE PERIODS IS INDICATED BY ARROWHEADS.

The form of the current variation is determined by the generalized angular velocity; the magnitude is determined by the coefficient I, and the phase, by the angle, θ.

It is interesting to note that the current is passing through a maximum value at the moment that the current vector makes an angle arc tan α/ω with the projection axis. This occurs when the vector is in such a position that the angular velocity m, drawn from its terminus, is perpendicular to the projection axis. See Fig. 3. A steady sinusoidal current is maximum at the moment that the vector representing it lies in the projection axis.

The behavior of damped sinusoidal currents in electric circuits, having resistance inductance and capacitance, has been the subject of considerable investigation, and is familiar, especially to those who have been interested in radio communication. The field of application of this theory, however, is much broader than this, and may easily be extended to include practically all electric machinery, both alternating and direct-current.

Currents and pressures that have the same generalized angular velocity can, of course, be combined by the customary principles of vector addition. The resulting vector diagrams will then be similar to those which apply to the steady state, save that the electromotive forces due to induction, both self and mutual, are not in time quadrature with the current, as we shall presently see. It is further understood that, as they rotate, all of the vectors shrink exponentially.

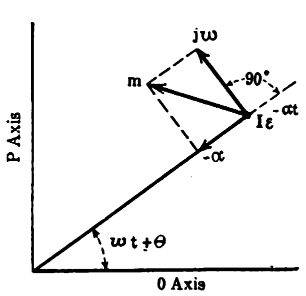

FIG. 2

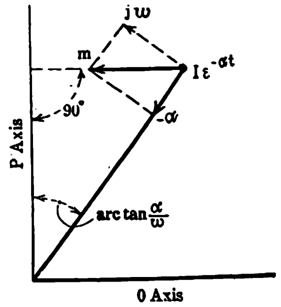

FIG. 3

Inasmuch as those who are interested in the operation of electric machinery have had little contact with the theory of damped sinusoidal currents, it may be well to show how these currents and the electric pressures they produce can be represented by vectors. In all of the following analyses any capacitance that the circuit may possess will be disregarded. This, of course, places a definite limitation upon the scope of the analysis and prevents the discussion of certain transient conditions, particularly those of relatively high frequency, which are very important in transmission circuits.

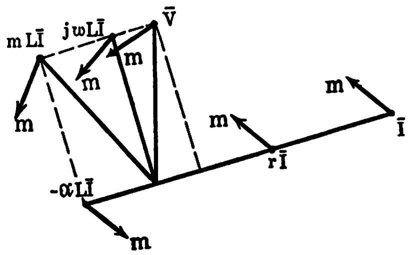

FIG. 4

When a current of the general form flows through a non-reactive resistance the *fall* in potential is $r\,I\,\epsilon^{-\alpha t}\sin(\omega\,t + \theta)$. This may be represented vectorially as in Fig. 4. The *fall* in potential through an inductance is:

$$L\,\frac{d\,i}{d\,t} = -\,\alpha\,L\,I\,\epsilon^{-\alpha t}\sin(\omega\,t + \theta) + \omega\,L\,I\,\epsilon^{-\alpha t}\cos(\omega\,t + \theta)$$

The first of these components may be represented by a vector in opposition to the current vector, and the second by a vector which leads the current by 90 degrees.

See Fig. 4. In this figure and hereafter a line drawn above a letter indicates that it represents a current or pressure of the general form.

Thus: $\bar{V} = r\,\bar{I} + (-\alpha + j\,\omega)\,L\,\bar{I}$

or $\bar{V} = (r + m\,L)\,\bar{I}$

where m represents the generalized angular velocity of the current vector; the generalized impedance is $(r + m\,L)$.

In Fig. 4 notice that there is a *rise* in potential through the circuit, numerically equal to $\alpha\,L\,I$, which is in time phase with the current. This indicates that electric energy is being generated within the circuit, due to the diminution of the magnetic field. The power so generated, $\alpha\,L\,I^2$, partially supplies the copper loss, $I^2\,r$, in the circuit. The vector representing the fall in pressure due to self inductance leads the vector representing the current by arc tan α/ω more than 90 deg. It is in this respect that the vector diagrams representing the transient state differ from those that represent the steady state.

If the electric circuit is linked with a magnetic circuit in which eddy current and hysteresis losses may occur, it is customary, though strictly speaking it is incorrect, to represent both the current and the magnetic flux by vectors when they have a steady sinusoidal variation.

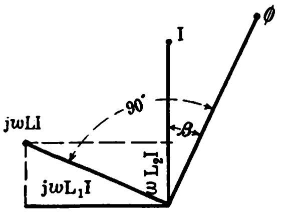

Fig. 5

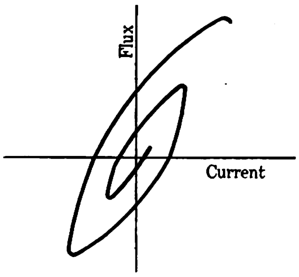

Fig. 6

In this case the flux is represented as lagging the current which produces it by a constant angle β. See Fig. 5. This is equivalent to assuming that the hysteresis loop is an ellipse and that the eddy currents are all in time quadrature with the flux in the core. The fall in reactive pressure $j\,\omega\,L\,I$, leads the flux, ϕ, by 90 deg., but the current by an angle which is somewhat less. This being the case, it is necessary to represent the inductance by a complex number, i. e. $L = L_1 - j\,L_2$; where $L_2/L_1 = \tan\beta$. Thus $j\,\omega\,L\,I = j\,\omega\,(L_1 - j\,L_2)\,I$

$$= \omega\,L_2\,I + j\,\omega\,L_1\,I$$

Similarly if the current is of the general form,—and we make the same assumption in regard to the constancy of the phase angle between the flux and the current,—the fall in pressure due to inductance is $(-\alpha + j\,\omega)\,(L_1 - j\,L_2)\,\bar{I}$, which may still be written $m\,L\,\bar{I}$, understanding now that both m and L are complex numbers. On expansion, this becomes $[\,(-\alpha\,L_1 + \omega\,L_2) + j\,(\omega\,L_1 + \alpha\,L_2)\,]\,\bar{I}$. If $\alpha\,L_1$ equals $\omega\,L_2$, i. e. if $\alpha/\omega = \tan\beta$, the fall in pressure due to inductance is in exact quadrature with the current and the energy given up by the magnetic field due to its collapse is just sufficient to supply the core losses. The power absorbed in the coil is then equal to the copper loss. If, with the same frequency, the rate of collapse of the magnetic field is greater than this, the coil absorbs less power than the copper loss.

If there is another electric circuit linking the same magnetic core, the fall in electric pressure through it produced by the action of the current in the first circuit is $j\,\omega\,M\,\bar{I}_1$. Notice that this is a vector which leads the current vector by more than 90 deg. When there are core losses in the mutual magnetic circuit the mutual inductance, M_1, might likewise be represented by a complex number of the form, $M_1 - j\,M_2$.

Only considerable research can determine the value of this suggestion that the self and mutual inductance be represented by complex numbers when the currents are decaying sinusoids. It should be noted that there is a constant lag angle of flux with respect to magnetizing current only if the hysteresis "loop" is a diminishing ellipse such as shown in Fig. 6. This is, of course, not true and whether it is justifiable can only be determined by experiment.

In any inductive circuit the current cannot immediately respond to an abrupt change in the magnitude or form of the applied electromotive forces. There is always a transient condition of current flow, the duration of which is determined by the inductance and resistance of the circuit. Moreover, the form of this transient flow is determined solely by these constants and is wholly independent of the character of the applied electromotive forces. The magnitude and phase of the transient, however, are determined by the initial and final values of the current in the circuit, and these are fixed by the applied electromotive forces.

TWO CIRCUITS HAVING MUTUAL INDUCTANCE BUT NO MOTION WITH RESPECT TO EACH OTHER

The simplest case we shall consider is that of a single-phase transformer which is simultaneously short-circuited on both primary and secondary sides.

With the primary winding short-circuited the fall in pressure through it is zero.

$$\bar{I}_1\,r_1 + \bar{I}_1\,m\,L_1 + \bar{I}_2\,m\,M = 0 \qquad (1)$$

r_1 and L_1 represent the resistance and self inductance, not leakage inductance, of the primary. M is the mutual inductance between primary and secondary. Likewise:

$$\bar{I}_2\,r_2 + \bar{I}_2\,m\,L_2 + \bar{I}_1\,m\,M = 0 \qquad (2)$$

The primary and secondary currents have the same form and thus the same generalized angular velocity. Eliminating $\bar{I}_1$ and $\bar{I}_2$ from equations (1) and (2) gives:

$$(r_1 + m\,L_1)\,(r_2 + m\,L_2) = m^2\,M^2$$

Expand and divide each term by $L_1\,L_2$ giving,

$$\sigma\,m^2 + (k_1 + k_2)\,m + k_1\,k_2 = 0 \qquad (3)$$

where $\sigma = 1 - \dfrac{M^2}{L_1\,L_2}$ $\quad k_1 = r_1/L_1$ and $k_2 = r_2/L_2$

By some writers σ is called the leakage coefficient, whereas others define the leakage coefficient as $\frac{L_1 L_2}{M^2} - 1$. In the latter case $\left(\frac{L_1 L_2}{M^2} - 1\right) = \sigma' = \frac{\sigma}{1-\sigma}$; or $\sigma = \frac{\sigma'}{1+\sigma'}$. In a transformer in which the resistance of the windings may be neglected this leakage coefficient, σ, is the ratio of the current on open circuit to that on short circuit with the same applied potential. k_1 and k_2 are the reciprocals of the time constants of the two windings on open circuit.

The solution for the generalized angular velocity is

$$m = -\frac{k_1 + k_2}{2\sigma} \pm \frac{\sqrt{(k_1 + k_2)^2 - 4\sigma k_1 k_2}}{2\sigma} \qquad (4)$$

Since σ is less than unity, the values of m are both real, and the transient current consists of two direct-current components one of which diminishes much faster than the other. These currents may be represented by vectors as in Fig. 1B.

In this case, in which both windings are short-circuited, there will be no current in either winding after the transient has disappeared. Thus the only current after short-circuit occurs is the transient. Let the vectors I_{10} and I_{20} represent the primary and secondary currents before short circuit occurs. Also let $\bar{I}_1'$ and $\bar{I}_1''$ represent the components of the transient current in the primary and similarly for the secondary. Then at the moment of short circuit, the sum of the instantaneous values of the two transient components must equal the instantaneous value of the initial current in both primary and secondary. The transient components might thus be represented by an indefinitely great number of vectors since it is only necessary that their projection on the given axis have specified values. It is more convenient, however, if they are chosen so that their vector sum is equal to the vector representing the initial current before short circuit. Thus we will write:

$$I_{10} = \bar{I}_1' + \bar{I}_1'' \qquad (5)$$

and

$$I_{20} = \bar{I}_2' + \bar{I}_2'' \qquad (6)$$

Remember that the vectors $\bar{I}_1'$, etc., are diminishing exponentially and that this equality holds only at the moment of short circuit. Further, notice that the vectors I_{10} and I_{20} are drawn to represent the *maximum* values of the currents before short circuit. Each of these transient components must satisfy equations (**1**) and (**2**). Substitute in equation (**1**) and write in the following form:

$$\bar{I}_1' \frac{(r_1 + m' L_1)}{m' M} = -\bar{I}_2'$$

also

$$\bar{I}_1'' \frac{(r_1 + m'' L_1)}{m'' M} = -\bar{I}_2''$$

where m' and m'' are the two values of m as indicated by equation (**4**). If these equations are added and $(I_{10} - \bar{I}_1')$ is substituted for $\bar{I}_1''$ the solution for $\bar{I}_1'$ is:

$$I_1' = \frac{m' m''}{k_1 (m' - m'')} \{ (k_1/m'' + 1) I_{10} + \sqrt{1-\sigma}\sqrt{L_2/L_1}\, I_{20} \} \qquad (7)$$

also

$$I_1'' = \frac{m' m''}{k_1 (m'' - m')} \{ (k_1/m' + 1) I_{10} + \sqrt{1-\sigma}\sqrt{L_2/L_1}\, I_{20} \} \qquad (8)$$

The product of the two roots divided by their difference is

$$\frac{m' m''}{m' - m''} = \frac{k_1 k_2}{\sqrt{(k_1 + k_2)^2 - 4\sigma k_1 k_2}}$$

In a transformer the ratio of the self-inductances of the primary and secondary is practically equal to the square of the ratio of the turns. Thus as is commonly said, $\sqrt{L_2/L_1}\, I_{20}$ is the secondary current referred to the primary. In order to reduce these expressions to a more understandable form, we will assume that k_1 equals k_2. This is very nearly true. If this is done the two components of the primary current become:

$$\bar{I}_1' = 1/2\,(I_{10} + I_{20}) \qquad (9)$$

$$\bar{I}_1'' = 1/2\,(I_{10} - I_{20}) \qquad (10)$$

where I_{20} is now the secondary current referred to the primary. The values of m reduce to:

$$m' = -\frac{r}{L + M}$$

$$m'' = -\frac{r}{L - M}$$

where r and L are the resistance and self-inductance of the primary winding, and M is the mutual inductance on the assumption that both windings have the same number of turns. In this case the difference of the self and mutual inductances is the leakage inductance of one winding. The two components of the transient secondary current are similar to those of the primary. Thus at the moment of short circuit each of the currents in the two windings breaks up into two components, one of which is one-half of the no-load current; $(I_{10} + I_{20})$ is the no-load current. This component decays at a rate determined by the resistance and the sum of the self and mutual inductances of the two windings. It is due to the energy stored in the magnetic core of the transformer. Its rate of decay is much slower than that of the other component. The latter's rate of decay is determined by the resistance and leakage inductance of the windings, and this component of the currents is due to the energy stored in the leakage field. The vector diagram and oscillograph record are given in Fig. 7. In this figure the vectors I_1 and I_2, representing the primary and secondary currents, are rotating at an angular velocity of ω. At the moment of short circuit each of these vectors abruptly stops and breaks up into two component vectors which diminish at different rates.

The more usual transient condition would be one in which the short circuit occurred on the secondary side alone. In this case the current immediately after short circuit consists of the transient current together with the steady value of the short-circuit current. The conditions that must be satisfied are now:

$$I_{10} = \bar{I}_1' + \bar{I}_1'' + I_{sc1} \qquad (11)$$

and

$$I_{20} = \bar{I}_2' + \bar{I}_2'' + I_{sc2} \qquad (12)$$

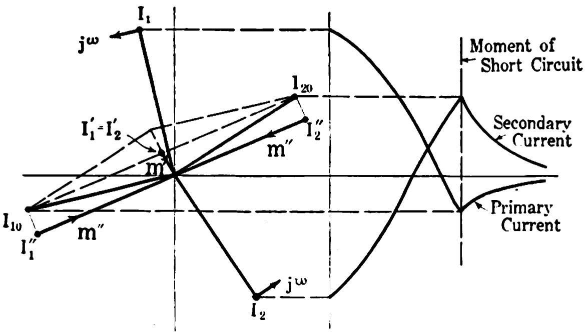

FIG. 7

where I_{sc1} and I_{sc2} represent the steady values of the short-circuit currents in the primary and secondary windings. As before this is a vector equation which holds only at the moment of short circuit. If these steady short-circuit currents are transferred to the left of the equality sign it is readily seen that the values of the components of the transient currents become:

$$I_1' = 1/2\,(I_{10} + I_{20}) - 1/2\,(I_{sc1} + I_{sc2}) \qquad (13)$$

$$I_1'' = 1/2\,(I_{10} - I_{20}) - 1/2\,(I_{sc1} - I_{sc2}) \qquad (14)$$

As before the secondary components are similar to the primary components. It is now necessary to determine the phase relation between the steady short-circuit current and the current in the transformer before the short circuit occurs.

In order to keep the diagrams as simple as possible, we will make the customary assumption that the primary and secondary currents are equal and in phase opposition. In that case the first component of the transient current is zero, and the second component is equal to the current before short circuit minus the steady short-circuit current. [See equations (13) and (14)] with $I_{10} = -I_{20}$ and $I_{sc1} = -I_{sc2}$. The error thus introduced is usually very slight. The vector diagram at the moment of short circuit is given in Fig. 8.

Before short circuit occurs only the four vectors representing the primary and secondary potentials and current appear. They are rotating at a uniform angular velocity, ω. At the instant after short circuit all of these except the primary potential disappear and in their place spring the steady short-circuit currents. I_{sc1} and I_{sc2} and the transient currents I_1'' and I_2''. The former continue to rotate at an angular velocity of ω without change in magnitude, while the latter remain fixed in angular position but diminish exponentially at a rate, α, depending upon the resistance and leakage inductance of the windings. The arrows at the ends of the vectors indicate the direction of their motion. Observe that, at the moment of short circuit, the current before short circuit is the vector sum of the steady short-circuit current and the transient current. Since the steady short-circuit current is from 10 to 25 times the full-load current, it makes little difference whether the short circuit comes at a time of full load or no load. It may be a little more severe at no load. On account of the excessive value of the steady short-circuit current, the vectors representing *it* and the *transient current* are essentially in phase opposition at the moment of short circuit. Half a cycle after short circuit occurs the vector I_{sc1} will thus have overtaken the stationary vector I_1'' and their vector sum will be a maximum. It at once follows that short circuit is most severe when it occurs at a moment that the steady short-circuit current would be a maximum.[2] Since the steady short-circuit current and applied potential are nearly in quadrature, for example, the power factor of a certain 1000-kv-a., 60-cycle transformer is 0.15 at short circuit,—the worst moment for a short circuit to occur is at about the time that the applied potential is passing through zero. In the transformer just cited the ratio of the resistance to the leakage inductance is 56 and thus in the time of one half cycle the transient current will have fallen to $\epsilon^{-\frac{56}{120}}$ or 63 per cent of its initial value. Thus the first current rush would be to a value which is 63 per cent greater than the maximum value of the steady short-circuit current which is itself more than 20 times the maximum value of the full-load current. The

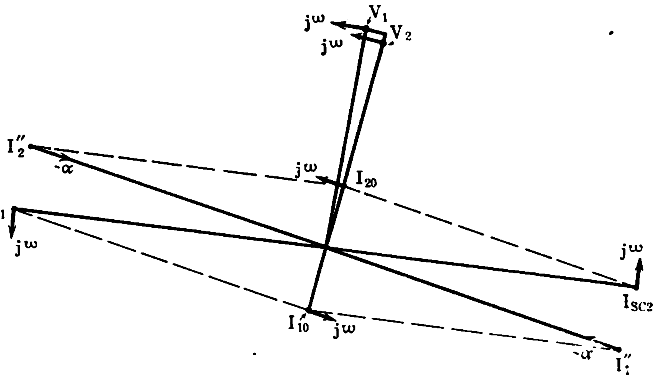

FIG. 8

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[2] To determine the exact moment at which the short-circuit current would be a maximum involves the solution of a mixed trigonometric and exponential equation, which is decidedly not worth while in the present problem.

forces between turns vary as the square of the current, and thus the first shock on this transformer may be 265 per cent of its ultimate maximum value.

When these forces have their maximum value they occur one-one-hundred-and-twentieth (1/120) of a second after short circuit in a sixty-cycle transformer and one-fiftieth (1/50) of a second after short-circuit in a twenty-five-cycle transformer. Nevertheless with transformers having the same short-circuit characteristics (volts, amperes, and watts on short-circuit test), the maximum forces that the windings must sustain are the same for all frequencies.

If the transformer is but a portion of the electric circuit in which the transient takes place the characteristic equations (1) and (2) must be written:

$$\bar{I}_1 r_1 + \bar{I}_1 m L_1 + \bar{I}_2 m M = \overline{V}_1 \qquad (15)$$

$$\bar{I}_2 r_2 + \bar{I}_2 m L_2 + \bar{I}_1 m M = \overline{V}_2 \qquad (16)$$

$\overline{V}_1$ and $\overline{V}_2$ are the transient potentials which appear at the terminals of the transformer. If the transformer has a relatively small exciting current little error will be made if it is treated like a simple inductive reactance. In this case the relation between the primary and secondary pressures is much simpler.

$$\overline{V}_1 - \bar{I}_1 (R + m L) = a \overline{V}_2 \qquad (17)$$

R and L are the equivalent resistance and leakage inductance of the two windings and a is the ratio of transformation at no load.

Two Circuits Having Mutual Inductance, in One of Which There is a Commutated Rotational E. M. F.

Before turning our attention to polyphase rotating machines, it might be well to consider briefly the direct-current dynamo of either the shunt or compound type. This apparatus is in some respects quite similar to the transformer. It has two circuits which are mutually inductive, the armature and series field forming one and the shunt field the other. There is this difference, however: an electromotive force is generated in the armature circuit by its rotation in a magnetic field that is due to the joint action of the current in both circuits. With constant speed this generated e. m. f., is proportional to the magnetic field. The relation between the magnetic field and the currents in the armature and shunt field circuits which produce it is similar to the well-known hysteresis loop, for which there is no simple mathematical expression. In some problems in which the exciting current is a steady alternating one, it has been found convenient, however, to assume that this hysteresis loop is an ellipse. If this assumption should prove sufficiently accurate in the present case also, it will simplify the problem greatly, for as has been already suggested, we may then represent the magnetic field and the rotational e. m. f. which it produces, together with the net exciting ampere-turns by vectors even when these quantities are damped sinusoids, *i. e.*, of the general form. This vector relation is shown in Fig. 9, which is drawn for the steady state.

The flux, ϕ, and the rotational e. m. f. E which is proportional to it, lag the exciting current I by a constant angle β. Thus we may write

$$E = (K_2 - j K_2) I \qquad (18)$$

$$\text{or } E = K I$$

where K is a complex number, $K_1 - j K_2$, having the dimensions of resistance. The vector equation shows that when the instantaneous value of the current is zero the generated e. m. f. is $K_2 I$. This is generally referred to as the residual voltage. At the moment that the current has its greatest value the rotational e. m. f. is $K_1 I$. If the exciting current is going through a steady cyclic variation, it is a simple matter to determine these two components of the e. m. f. See Fig. 10. When the current is a damped sinusoid, however, the vector representing it is shorter at the moment that it is perpendicular to the time axis than it was earlier in the cycle when the current was passing through its maximum value. See Fig. 11. In this case the best method of determining the constants K_1 and K_2, in order to assure the greatest accuracy in the calculations can only be determined by experiment.

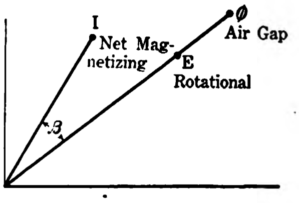

Fig. 9

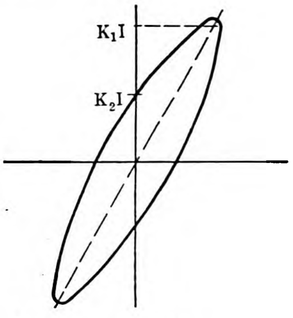

Fig. 10

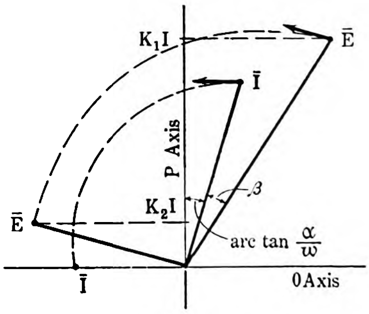

Fig. 11

Let $\bar{I}_1$ and $\bar{I}_2$ be the currents in the armature and shunt field circuits and similarly, r_1 and L_1, the resistance and self-induction of the one, and r_2 and L_2, the same constants of the other. Also let M be the mutual inductance between the two circuits. The e. m. f., $\bar{E}$, generated in the armature circuit by rotation is:

$$\bar{E} = K (N_1/N_2 \bar{I}_1 + \bar{I}_2);$$

where N_1 and N_2 are the effective turns in each circuit acting on the main magnetic axis. Let $N_1/N_2 = a$. Also let $\bar{V}$ be the rise in terminal pressure across both the armature and shunt field circuits.

Then we have:

$$\bar{V} = \bar{E} - \bar{I}_1 r_1 - \bar{I}_1 m L_1 - \bar{I}_2 m M \quad \text{(armature circuit)}$$

$$= \bar{I}_1 (K a - r_1 - m L_1) + \bar{I}_2 (K - m M) \quad \textbf{(19)}$$

also

$$\bar{V} = - \bar{I}_2 r_2 - \bar{I}_2 m L_2 - \bar{I}_1 m M \quad \text{(shunt field circuit)}$$

$$= - \bar{I}_2 (r_2 + m L_2) - \bar{I}_1 m M \quad \textbf{(20)}$$

If these equations, (**19**) and (**20**), are solved for $\bar{I}_1$ and $\bar{I}_2$ we have:

$$\bar{I}_1 = \frac{r_2 + m L_2 + K - m M}{(K a - r_1 - m L_1)(r_2 + m L_2) - (K - m M) m M} \bar{V}$$

and

$$\bar{I}_2 = - \frac{K a - r_1 - m L_1 + m M}{(K a - r_1 - m L_1)(r_2 + m L_2) - (K - m M) m M} \bar{V}$$

The total net current delivered, $\bar{I} = \bar{I}_1 + \bar{I}_2$, is

$$\bar{I} = \frac{r_1 + r_2 + K (1 - a) + m (L_1 + L_2 - 2 M)}{(K a - r_1 - m L_1)(r_2 + m L_2) - (K - m M) m M} \bar{V}$$

The ratio of the fall in terminal potential, $-\bar{V}$, to the total current is the generalized impedance, Z, of the dynamo. Thus we have:

$$Z = \frac{(K - m M) m M - (K a - r_1 - m L_1)(r_2 + m L_2)}{r_1 + r_2 + K (1 - a) + m (L_1 + L_2 - 2 M)}$$

For further discussion of this problem the reader is referred to the paper by Doherty on "Exciter Instability" presented at the Pacific Coast Convention, August, 1922.

Two Systems of Balanced Polyphase Circuits Having a Constant Relative Angular Velocity

The author believes that the simplest method of presenting the analysis of the transient condition in polyphase induction and synchronous machines is that which considers the rotating magnetic fields in the air gap and the reactions which they produce. When these polyphase machines are operating under balanced conditions, either steady or transient, the currents in the several phases of the polyphase winding may be represented by vectors of equal length equally spaced with respect to each other. For the steady condition the vectors are of constant length and rotate at a constant angular velocity. This representation has been used for many years. The *only* difference between the steady and the transient condition is that the vectors which represent the latter state diminish exponentially as they rotate. If this conception of the transient state is borne in mind the following analysis is readily followed by anyone who is familiar with the steady operation of these machines. In all that follows the relative speed of the two windings is assumed to remain constant throughout the time considered. Since the maximum current occurs within about a half-cycle after the transient begins the angular velocity of the moving part has little opportunity to change before the first shock is over.

The parallel lines in Fig. 12 represent the uniform air gap of a three-phase induction motor. The phase belts of the stator conductors are indicated at the letters a, b, c. The pole pitch is from $+ a$ to $- a$. If this winding is properly designed, there will be very small harmonics in the space distribution of the air-gap flux density when the alternating currents are balanced. We will neglect whatever harmonics may exist and assume that this distribution may be represented by a sinusoidal curve such as β. The line $X - X$ is the axis of this curve. If the current in phase b lags that in phase a by 120 deg. this magnetic field will move toward the right at a constant angular velocity of ω radians per second; where ω equals the frequency of the alternating current multiplied by 2π. On the other hand, if the current in phase b leads that in phase a the field moves toward the left at the same velocity. In all that follows we shall consider that a velocity toward the right is positive and that one toward the left is negative. Furthermore, the magnitude of the field as well as its form, will not change as it moves, and thus the field might be represented by a rotating vector whose projection on a fixed axis indicates the amount of flux linking any particular phase. The angular displacement of this vector from the fixed axis is always the angle between the axis of the moving field and the axis of the phase considered. See Fig. 13. One of the most important points in the theory of rotating fields that are produced in this manner in a uniform air gap is that the flux through any phase is a maximum at the moment that the current in that phase is also a maximum. For example, when the current in phase a is maximum the field is in the position β_a. One sixth of a cycle later, the current in $- c$ will be a maximum and in the same time interval, the flux distribution will have moved one sixth of two full pole pitches toward the right and be a maximum through this phase, $- c$. In a similar manner, it will have moved into the position β_b by the time that the current in phase b is a maximum. This can be represented vectorially as in Fig. 13. The current and flux are

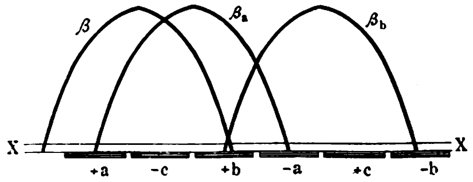

Fig. 12

represented by the rotating vectors I and φ. The instantaneous value of the current in any phase or the flux through that phase is the projection of the corresponding vector on the axis of that phase. In Fig. 13 the vectors are drawn for the time that the current in phase $-c$ has its maximum value. Notice that for both the phases a and b the currents in them and the fluxes through them are each one half of their maximum values. There is this difference, however, for while the values are decreasing for phase a they are increasing for phase b, in which they will be a maximum 1/6 of a cycle later. Ordinarily, we are content to know the current and flux in any one phase, a, for example, and then the projection axes for the other phases are not drawn.

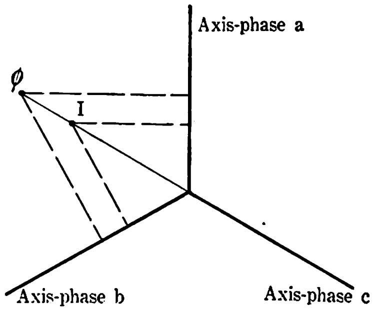

Fig. 13

It may be convenient and possibly necessary in some problems to consider magnetic fields that rotate to the left, that is, with a negative angular velocity. In this case the vector, φ, representing the flux through any phase would likewise have a negative angular velocity and would thus move in the clockwise direction. If the trigonometric expressions for the instantaneous currents in the phases are written it will be noticed that giving a negative value to ω is equivalent to reversing the phase order of the currents. Such a reversal of phase order reverses the direction of rotation of the magnetic field produced by the current. If the current vector in Fig. 13 is rotated in the negative or clockwise direction it will be noticed that the current in phase b will lead that in phase a by one-third of a cycle, that is the phase order of the currents is reversed. We must be most careful, however, to remember that, if it is necessary to combine or operate on these vectors, whether they have positive or negative angular velocities the operator j *always* indicates a phase rotation of one quadrant in the positive or counter clockwise direction. If this is not done we may become hopelessly confused. If the air gap is not uniform, as is the case with salient pole machines, the curve of flux density will alter both its shape and magnitude as it rotates, but that is a complication which it is well for the present to disregard, leaving its consideration until some future time.

In general if the currents in the several phases of a symmetrical n-phase winding have values equal to the projections of a "current" vector on each of a system of n equally spaced axes, the magnetic flux linking any winding is equal to the projection of a "flux" vector, which is drawn in phase with the "current" vector, on the axis corresponding to that winding. This assumes that the windings are so designed that there are essentially no harmonics in the space distribution of the magnetic field and that each winding will, if acting alone, produce exactly the same flux per ampere.

If the currents are of the general form both the "current" and "flux" vectors shrink exponentially as they rotate. The space distribution of the magnetic field is still sinusoidal at all times but its magnitude diminishes as it moves through the air gap. Compare Fig. 13 with Fig. 14. In the latter figure let us assume that the currents took on the general form at the moment that the flux distribution was at the point indicated by the curve β, and that their generalized angular velocity is $(-\alpha + j\,\omega)$. The equation of the maximum density for any later position of the field such as β_1 is: $B_m = B_o\epsilon^{-\alpha t}$; where t is the time required for the field to move from the position β to β_1, *i. e.*, through an angular distance X.

The flux through any phase as well as the current in it may still be represented by a vector although this vector is now of the general type—that is, it shrinks exponentially as it rotates. A current or flux variation of the general form reaches its maximum value in any phase at the moment that the vector which represents it is still arc tan α/ω from the projection axis of this phase. Because the field is shrinking as it rotates the flux linking phase a is greater when the field is in the position β_1 than when it is in the position β_2 at which moment the axis of the field is coincident with the axis of the phase. See Fig. 15. Thus the flux through any phase is a maximum at the moment that the current in the phase is a maximum, but the flux distribution is no longer directly opposite the phase when this occurs, as was the case when the currents were steady sinusoids.

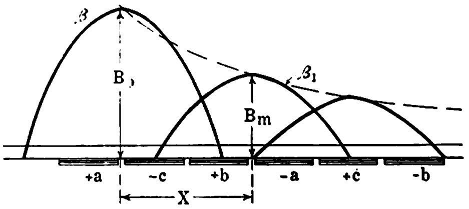

Fig. 14

It is then evident that even in the transient state the flux through any phase and the current in that phase may still be represented by coincident vectors.

The electromotive force generated by this rotating field is the negative of the time rate of change of the flux linkages. Thus, in general, when this time rate of flux change in any phase is zero the e. m. f. is zero and the flux linking the phase is a maximum. In Fig.

16 the flux in phase a is a maximum when the vector representing it is in the position shown. At this moment the e. m. f. is zero and the vector, $\bar{E}$, representing it must, therefore, be perpendicular to the projection axis. An additional proof of this is interesting even if unnecessary. Consider Fig. 15. When the field is in the position β_1 the e. m. f. produced in the phase a has two components, one due to the motion of the field and the other due to its shrinking. The motional e. m. f. is proportional to the velocity of the field and to the sine of the angle, ωt, between the field and the phase winding. If at the point $+a$, the direction of the flux density is upward and the motion of the field is to the right the e. m. f. generated in the belt, $+a$, is away from the observer. On the other hand if the field within a circuit changes an e. m. f. is set up which is proportional to the rate of change and is in such a direction as to oppose the change. The e. m. f. thus produced is proportional to $\alpha \cos \omega t$, and is toward the observer in the belt of conductors $+a$. If the angular displacement, ωt, is so chosen that the resulting e. m. f. is zero, $\omega \sin \omega t$ equals $\alpha \cos \omega t$. That is to say, the angular displacement of the flux distribution is arc tan α/ω. Observe that the fall in potential, $-\bar{E}$, leads the flux, $\bar{\phi}$, by an angle, $\pi/2 + \text{arc tan } \alpha/\omega$, just as does the fall in potential due to self induction. Thus, in a polyphase winding we may combine the fall in pressure due to the so-called leakage flux in one phase with the fall in pressure in this phase due to the mutual air-gap flux which is produced by all phases. This is commonly done in the analysis of synchronous machines and the resulting loss in pressure is the so-called synchronous inductance, or reactance drop. If L represents the synchronous inductance, a transient current, $\bar{I}$, having a generalized angular velocity m, will produce a fall in potential of $m L\bar{I}$. This fall in potential is represented by a vector which is $\pi/2 + \text{arc tan } \alpha/\omega$ ahead of the current vector.

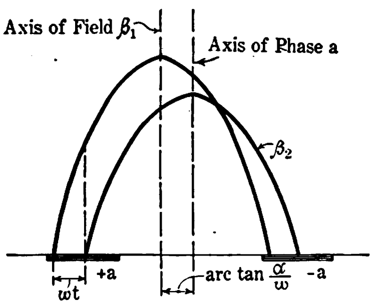

FIG. 15

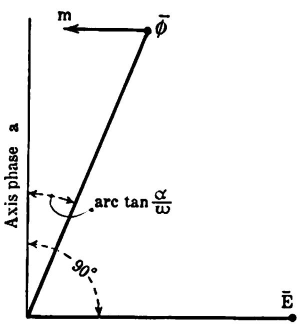

FIG. 16

This conception of the action of a rotating field of the transient type may be summarized as follows: Consider two groups of symmetrical polyphase windings, a and b, the first of which is stationary and the second, moving in the positive direction at an angular velocity of p radians per sec. The projection axis for the stationary winding is stationary, while the projection axis for the moving winding is moving at the same angular velocity, *viz.* p, as indicated in Fig. 17. If the angular velocity of the currents in the stationary, or a, winding, is ω radians per sec., the relative angular velocity of the magnetic field due to these currents with respect to the b winding is $(\omega - p)$ radians per sec. in the positive direction. The flux linking any phase of the b winding is a maximum arc tan $\frac{\alpha}{\omega - p}$ before the flux vector lies in the projection axis of this phase. If the field is moving with respect to the b winding in the positive direction the flux will be a maximum when the flux vector is at $\bar{\phi}_a'$. See Fig. 17. If the field is moving relatively in the negative direction, *i. e.* if p is greater than ω, the flux will be a maximum when the flux vector is at $\bar{\phi}_a''$. At the moment that the flux is a maximum through any phase, the generated e. m. f. in this phase is zero and the vector representing it will be at E_b' for a positive relative angular velocity and at E_b'' for a negative velocity. This generated e. m. f. is equal to the product of the current in one phase of the a windings, the relative generalized angular velocity with respect to the b windings of the field that these currents produce, and the mutual inductance between the a and b windings. The mutual inductance M can be measured as follows: Fix the a and b windings with respect to each other and send balanced polyphase currents through either, the a for windings for example. The mutual inductance M is the e. m. f. generated in one phase of the other, or b winding, divided by the product of the current in one phase of the a winding and angular velocity of this current. By the time that the flux has advanced $\pi/2 - \text{arc tan } \frac{\alpha}{\omega - p}$ with respect to the b windings, the generated e. m. f. will have reached its maximum value, but the current will usually not reach its maximum value until somewhat later depending

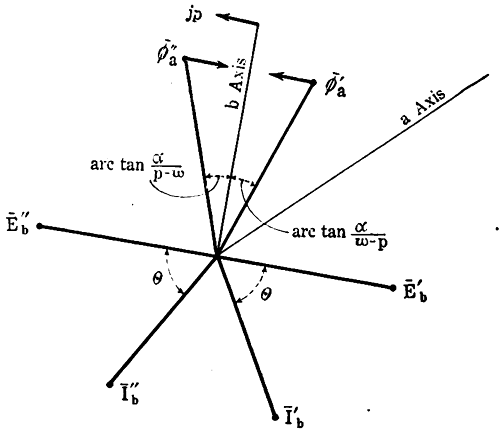

FIG. 17

upon the constants of the b windings and their terminal pressure. In any case, however, the current will be represented by a vector equal to $\frac{\bar{E}_b - \bar{V}_b}{r_b + m\,L_b}$; where $\bar{V}_b$ is the terminal pressure, r_b the resistance and L_b the synchronous inductance of one of the b phases. By the time the current in the winding has reached its maximum value the field will have advanced a further relative angle of θ. Thus whether the motion of the a field is positive or negative, its action on the b winding will develop a field which will always lag the a field in its motion with respect to the b winding. Consider an induction machine in which the magnetic field due to the stator, a windings, rotates in the positive direction at ω radians per sec. If the rotor turns in the same direction at less than this synchronous speed the relative motion of the stator field with respect to the rotor is positive. The rotor component of magnetic field actually and apparently lags the stator field. If the rotor turns in the same direction at a speed greater than synchronous, the relative motion of the stator field with respect to the rotor is negative and the rotor component of field apparently leads when observed from the fixed stator, though it actually lags, as would be observed from any point on the rotor itself. It is interesting that these well-known facts should apply to the transient state as well as to the steady operating condition.

Having established these general principles in regard to rotating fields during the transient state the solution for the transient currents in an induction motor may be written down without further explanation. If the generalized angular velocity of the stator currents is m the relative generalized angular velocity of the field produced by these currents with respect to the rotor is $(m - j\,p)$. Since $m = -\alpha + j\,\omega$ and the relative angular velocity of the stator magnetic field and the rotor winding is $(\omega - p)$, the generalized angular velocity of the rotor currents is $-\alpha + j\,(\omega - p)$, *i. e.*, $m - j\,p$. The generalized angular velocity of the rotor currents is thus $m - j\,p$, but the relative angular velocity with respect to the stator of the field produced by them is $[m - j\,p - (-j\,p)]$, *i. e.*, m. The second $(-j\,p)$ is the relative angular velocity of the stator with respect to the rotor. Thus we have:

$$\bar{E}_2 = -(m - j\,p)\,M\,\bar{I}_1 \tag{19}$$

$$\bar{E}_1 = -m\,M\,\bar{I}_2 \tag{20}$$

$\bar{E}_2$ and $\bar{E}_1$ are respectively the generated e. m. f. due to the stator currents in the rotor and that due to the rotor currents in the stator. The rotor current, $\bar{I}_2$, is:

$$\bar{I}_2 = \bar{E}_2/Z_2 \tag{21}$$

where Z_2 is the generalized self impedance of the rotor calculated for a generalized angular velocity of $m - j\,p$. The stator current $\bar{I}_1$, is:

$$\bar{I}_1 = \frac{\bar{E}_1 - \bar{V}_1}{Z_1} \tag{22}$$

where Z_1 is similarly the generalized self impedance of the stator calculated for a generalized angular velocity of m, and $\bar{V}_1$ is the transient terminal pressure of the stator. Combining these four simple relations gives the characteristic equation for the stator current of an inductor motor during the transient period. That is:

$$\bar{I}_1 = -\frac{Z_2}{Z_1\,Z_2 - m\,(m - j\,p)\,M^2}\,\bar{V}_1 \tag{23}$$

$$= \frac{-\bar{V}_1}{Z_T}$$

where $-V_1$ is the fall in terminal pressure across one phase of the stator. That is, it may be said that the transient impedance of an induction machine is:

$$Z_T = \frac{Z_1\,Z_2 - m\,(m - j\,p)\,M^2}{Z_2} \tag{24}$$

The steady operating impedance is

$$Z_0 = \frac{Z_1\,Z_2 - j\,\omega\,(j\,\omega - j\,p)\,M^2}{Z_2} \tag{25}$$

where $Z_1 = r_1 + j\,\omega\,L_1$ and $Z_2 = r_2 + j\,(\omega - p)\,L_2$. If the transient occurs when the stator winding is symmetrically short-circuited or if it occurs when the stator is connected to a circuit whose resistance and inductance are many times smaller than the similar constants of the motor, the transient terminal pressure of the stator is essentially zero. It is thus evident that the transient impedance is also zero. Equating the transient impedance Z_T to zero gives the characteristic equation for an induction motor and determines the generalized angular velocity of the stator current.

Since $Z_1 = r_1 + m\,L_1$ and $Z_2 = r_2 + (m - j\,p)\,L_2$ equating the transient impedance to zero gives

$$(r_1 + m\,L_1)\,[r_2 + (m - j\,p)\,L_2] - m\,(m - j\,p)\,M^2 = 0$$

Expanding and collecting terms and dividing by $L_1\,L_2$ gives the following quadratic in m:

$$\sigma\,m^2 + (k_1 + k_2 - j\,p\,\sigma)\,m + k_1\,k_2 - j\,p\,k_1 \tag{26}$$

where $k_1 = r_1/L_1$, $k_2 = r_2/L_2$ and $\sigma = 1 - \frac{M^2}{L_1\,L_2}$

Notice that this characteristic equation for the induction motor is identical with that for a transformer with the exception of two terms, $-j\,p\,\sigma\,m$, and $-j\,p\,k_1$. These terms, however, completely alter the form of the transient. The solution for the generalized angular velocity, m, is:

$$m = -\frac{(k_1 + k_2)}{2\,\sigma} + \frac{j\,p}{2} \pm \frac{\sqrt{(k_1 + k_2 - j\,p\,\sigma)^2 - 4\,\sigma\,(k_1\,k_2 - j\,p\,k_1)}}{2\,\sigma} \tag{27}$$

If $k_1 = k_2$, the quantity under the radical is real and will be negative unless the equivalent resistance is somewhat greater than the equivalent reactance as measured with the rotor blocked, a most unusual condition except possibly in very small motors. Furthermore the decrement $\frac{k_1 + k_2}{2\,\sigma}$ is very nearly equal to the resistance of the stator winding divided by the equivalent inductance, L_0, as determined from locked saturation test. Thus under ordinary conditions the two values of the generalized angular velocity are approximately:

$$m' = -\,r_1/L_0 + j\,p \qquad (28)$$

$$m'' = -\,r_1/L_0 + j\,O \qquad (29)$$

These are the values given by Doherty and Williamson. The magnitude of the error is best shown by a numerical example. The relative constants of a 25-cycle, 2600-h. p. induction motor are $r_1 = r_2 = L_1 = L_2$ and $\sigma = 0.075$. The equivalent leakage inductance as calculated from these values is relatively $L_0 = 0.0751$. The approximate values of the generalized angular velocity for a speed of 157 radians per sec. are:

$$m' = -\,13.31 + j\,157$$

$$m'' = -\,13.31 + j\,O$$

The values as calculated from the solution of the characteristic equation are:

$$m' = -\,13.33 + j\,155.8$$

$$m'' = -\,13.33 + j\,1.2$$

The error in the approximate values in this case at least is very small.

The form of the generalized angular velocity shows that the transient current during short circuit consists of two components, one having a frequency slightly below that corresponding to the speed of the motor, and the other a very low frequency. Notice that the sum of these frequencies always equals that corresponding to the speed of the motor. If the ratios of resistance to synchronous self inductance are not the same for the stator and rotor windings the components of the transient current will decay at different rates, otherwise their rates of diminution are ordinarily equal. During this transient short circuit the angular velocity of the rotor is greater than the angular velocity of the stator currents so that the machine is operating as an induction generator. In this case, the *sum* of the angular velocities of the corresponding stator and rotor component currents is equal to the angular velocity of the rotor. In calculating the performance of an induction machine under steady operating conditions it is customary to use an effective resistance of the stator winding and a true, or ohmic, resistance of the rotor winding. Similarly, it may prove more accurate to use an effective resistance of stator and a true resistance of rotor when calculating m', and, vice versa, a true resistance of stator and an effective resistance of rotor when calculating m''. If, for example, the ratio of effective to true resistance for both stator and rotor is 1.5, the rate of decay of the transient currents is about 25 per cent greater,[3] than would be calculated by using the true resistances. Again, if the ratios of true resistance to synchronous self inductance are the same for both stator and rotor, the rates of decay of the two transient components would be slightly different if effective resistances are substituted for true resistances as suggested.

Having determined the form of the transient it remains only to determine its magnitude. The fundamental principle is that the current in any inductive circuit cannot abruptly change its value. Thus, if the currents in the stator and rotor windings immediately before the transient occurs are represented by the vectors I_{10} and I_{20} in Fig. 18 the vector sum of the currents in these windings immediately after the transient begins must be respectively I_{10} and I_{20}. The case is somewhat different from the single-phase transformer inasmuch as there are now two or more stator or primary circuits and a corresponding number of rotor or secondary circuits in which the instantaneous values of the currents immediately before and after the transient begins must be the same. It has already been pointed out that for any instantaneous values of balanced polyphase currents in symmetrical windings both the currents and the magnetic field which they produce may be represented by *vectors* in phase with each other. If the currents in the stator and rotor windings do not abruptly change at the moment that the transient begins, it is evident that the component magnetic fields they produce in the air gap do not change in magnitude or in position. That is, the vectors which represent the component fields produced by the stator and rotor currents do not change at the moment that the transient begins. Since the current vectors are in phase with the flux vectors they also do not change abruptly. The general relation that holds at the moment that the transient begins is:

$$I_{10} = \bar{I}_1' + \bar{I}_1'' + I_{1s} \qquad (30)$$

$$I_{20} = \bar{I}_2' + \bar{I}_2'' + I_{2s} \qquad (31)$$

where I_{10} and I_{20} are the vectors representing the stator and rotor currents before the transient begins; $\bar{I}_1'$ and $\bar{I}_1''$ are the generalized vectors representing the two transient components of the stator currents; similarly $\bar{I}_2'$ and $\bar{I}_2''$ are the generalized vectors representing the transient components of the rotor current; I_{1s} and I_{2s} are the vectors which represent the steady currents in the stator and rotor which will exist alone after the transient has disappeared. As it is more convenient if the rotor currents are determined on the basis of a one to one ratio of transformation this will be done throughout. In this case $L_1 = L_2$, at least approximately. The steady operating conditions before and after the transient occurs are sufficient to determine the steady current vectors I_{10}, I_{1s}, etc. The following relations must hold for the transient currents:

$$\bar{I}_1' + \bar{I}_1'' = I_{10} - I_{1s}$$

$$\bar{I}_2' + \bar{I}_2'' = I_{20} - I_{2s}$$

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[3] A. I. E. E. Transactions, page 525.

If an induction motor or generator is short-circuited on all phases I_{1s} and I_{2s} are both zero since there will ultimately be no current in the machine. If an induction motor or generator is connected to a line when it carries no current, I_{10} and I_{20} are both zero. We shall work out in detail only the first of these cases since nothing is gained in generality by doing more.

Each of the transient components must satisfy the vector relation:

$$(r_1 + m\,L_1)\,\bar{I}_1 + m\,M\,\bar{I}_2 = O$$

and thus we may write:

$$(r_1 + m'\,L_1)\,\bar{I}_1{}' + m'\,M\,\bar{I}_2{}' = O \qquad (32)$$

$$(r_1 + m''\,L_1)\,\bar{I}_1{}'' + m''\,M\,\bar{I}_2{}'' = O \qquad (33)$$

also at the moment of short circuit

$$\bar{I}_1{}' + \bar{I}_1{}'' = I_{10} \qquad (34)$$

and

$$\bar{I}_2{}' + \bar{I}_2{}'' = I_{20} \qquad (35)$$

Solving these four simultaneous equations for the components of the transient currents in the stator winding gives:

$$I_1{}' = \frac{m'\,m''}{m' - m''}\,[M/r_1\,I_{20} + (1/k_1 + 1/m'')\,I_{10}]$$

and

$$\bar{I}_1{}'' = \frac{m'\,m''}{m'' - m'}\,[M/r_1\,I_{20} + (1/k_1 + 1/m')\,I_{10}]$$

These expressions are readily reduced to understandable forms if suitable approximations are made. If $k_1 = k_2 = k$ the values of m' and m'' are approximately

$$m' = -\,k/\sigma + j\,p \qquad (36)$$

$$m'' = -\,k/\sigma + j\,O \qquad (37)$$

Also

$$M = \sqrt{1 - \sigma}\,L$$
$$= (1 - \sigma/2)\,L \qquad \text{approximately}$$

where L is the synchronous self inductance of either the stator or rotor winding. The approximate values of the transient components become:[4]

$$I_1{}' = I_{10}/2 - 1/\sigma\,I_n$$

$$I_1{}'' = I_{10}/2 + 1/\sigma\,I_n$$

The error in these values is well within 10 per cent ordinarily. Notice that the transient currents depend almost entirely upon the no-load current, I_n, and the leakage coefficient, σ. It makes little difference whether short-circuit occurs at no-load or full load. The first of these components rotates at approximately synchronous speed while the second is practically stationary. Each shrinks exponentially at the same rate. The first maximum value of the transient current in any phase is greatest if it occurs at such a time that the transient components lie as nearly as possible in the projection axis for that phase at the moment that the transient begins. For, in that case, about one-half a cycle later the first component will have overtaken the second and the resulting projection on the axis of the phase will have its greatest value. During this time, however, each of the components will have shrunk to $\epsilon^{-\frac{k}{\sigma}\frac{\pi}{p}}$ of their initial values. The first maximum current rush may be to an approximate value of

$$\frac{2\,I_n}{\sigma}\,\epsilon^{-\frac{k}{\sigma}\frac{\pi}{p}} \qquad (38)$$

where I_n is the maximum value of the exciting current before the transient begins. This approximate solution checks with that given by Doherty and Williamson although the latter is in a slightly different form. The vector diagram is given in Fig. 18, which is drawn at the moment that the transient begins. The projection axes of one phase of the stator winding and of one phase of the rotor winding are shown. The former is fixed in position while the latter rotates at a speed

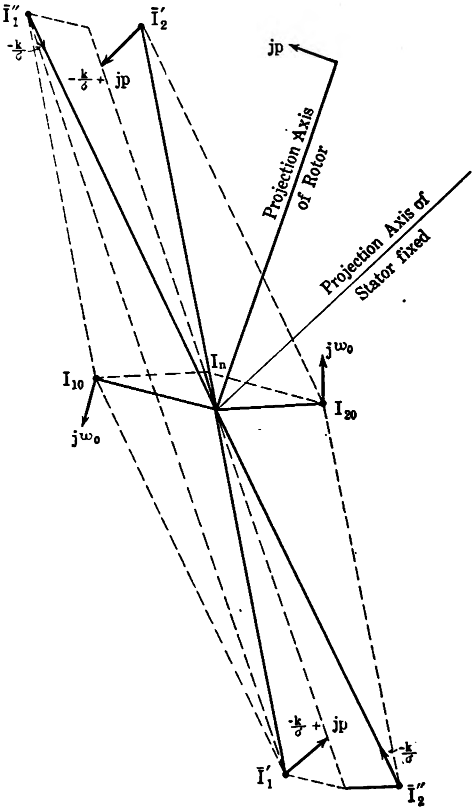

FIG. 18

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[4] More exact solutions for the component currents, based on the approximate values of m are:

$$\bar{I}_1{}' = \left(1 + j\,\frac{k}{\sigma\,p}\right)\left(I_{10}/2 - \frac{\sqrt{1-\sigma}}{\sigma}\,I_n\right)$$

an $I_1{}'' = \left(1 + j\,\frac{k}{\sigma\,p}\right)\left(\frac{1 - j\,\frac{k}{\sigma\,p}}{1 + j\,\frac{k}{\sigma\,p}}\,I_{10}/2 + \frac{\sqrt{1-\sigma}}{\sigma}\,I_n\right)$

equal to that of the rotor, *viz.*, p radians per sec. When s is the slip, p equals $(1 - s)\ \omega_0$. Before the transient begins only the vectors representing the stator and rotor currents appear. They are rotating at a uniform angular velocity, ω_0. At the moment of short circuit both of these vectors vanish and in their places spring the vectors which represent the transient currents. The arrows at the ends of these vectors represent their angular velocities.

Symmetrical Polyphase and Single-Phase Systems Moving with Respect to Each Other at a Constant Angular Velocity

The analysis of this problem is somewhat more difficult than that of the preceding. When a polyphase winding is acted upon by a rotating magnetic field it reacts and produces another magnetic field which always rotates at the same speed and in the same direction as does the first. This is not true of a single-phase winding, however, and it is herein that the difficulty lies. When a single-phase winding is excited by a rotating magnetic field, there is produced in it an alternating current of a frequency corresponding to their relative angular velocities. An alternating current in a single-phase winding produces a magnetic effect that is equivalent to two magnetic fields which rotate in opposite directions with respect to the winding at an angular velocity equal to that of the current in the winding. This well-known method of analysis is ascribed to Ferraris. The magnitude of each of these fields is one-half of the *maximum* value of the actual alternating field produced by the single-phase current.

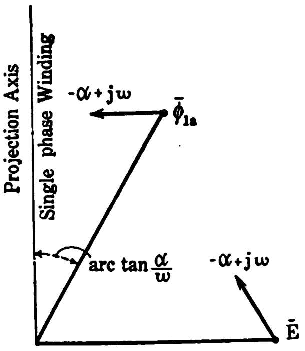

Fig. 19

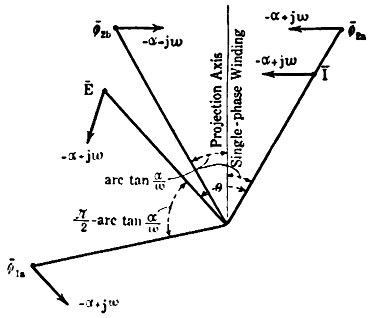

Fig. 20

We have already established the relation that exists between a rotating field and the field which may be produced by its action on a polyphase winding. It now becomes necessary to establish a corresponding relation when the rotating field acts on a single-phase winding. In Fig. 19 let $\overline{\phi}_{1a}$ represent a positively rotating magnetic field which has a complex angular velocity $(- \alpha + j\ \omega)$. The flux linking the single-phase winding will be a maximum at the moment when $\overline{\phi}_{1a}$ is in the position shown. The generated e. m. f., E, in the winding will be zero at this instant. By the time that $\overline{\phi}_{1a}$ has advanced $(\pi/2 - \text{arc tan}\ \alpha/\omega)$ this generated e. m. f. will be a maximum. The current however will not reach its maximum value until a time still later that is determined by the generalized self impedance of the single-phase winding. At the time that the current is a maximum the vector representing it is arc tan α/ω from the axis of the winding as shown in Fig. 20. At this moment one of the oppositely rotating fields $\overline{\phi}_{2a}$ is coincident with the current and the other, $\overline{\phi}_{2b}$, makes the same angle with but is on the other side of the axis of the winding. Of course, the axis of the field due to a single-phase current always coincides with the axis of the winding.

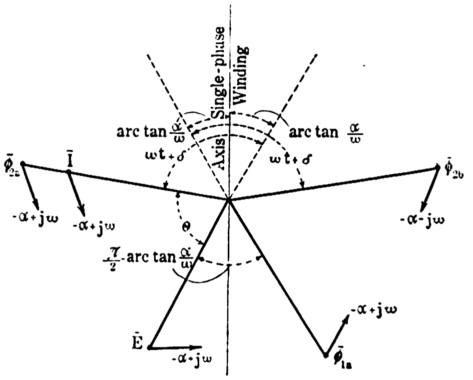
Axis Single-phase Winding
arc tan $\frac{\alpha}{\omega}$
ωt+δ
arc tan $\frac{\alpha}{\omega}$
ωt+δ
$\overline{\phi}_{2a}$
Ī
$\overline{\phi}_{2b}$
-α+jω
-α+jω
-α-jω
θ
$\frac{\pi}{2}$-arc tan $\frac{\alpha}{\omega}$
-α+jω
Ē
-α+jω
$\overline{\phi}_{1a}$

Fig. 21

One of these oppositely rotating fields travels at the same speed and in the same direction as does the initial field, $\overline{\phi}_{1a}$, while the other travels at the same speed but in the opposite direction. The condition at a time angle $\omega\, t + \delta$ after the current in the single-phase winding is a maximum is shown in Fig. 21. The oppositely rotating fields are represented by the vectors $\overline{\phi}_{2a}$ and $\overline{\phi}_{2b}$: the subscripts a and b indicate the direction of rotation. The constant phase relation between $\overline{\phi}_{1a}$ and $\overline{\phi}_{2a}$ will be determined presently.

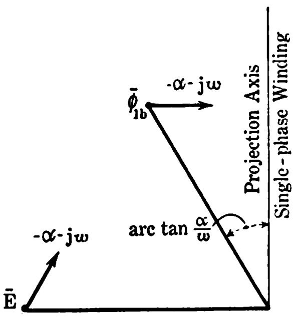

Fig. 22

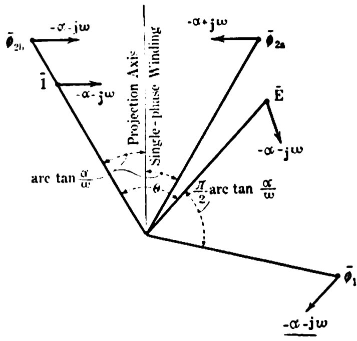

Fig. 23

If the first magnetic field had been rotating negatively with a complex angular velocity of $(- \alpha - j\ \omega)$ the flux linking the single-phase winding would have been maximum at the moment shown in Fig. 22. By the time the current had reached its maximum value this field would have reached the position indicated in

Fig. 23. The oppositely rotating components of the single-phase field are now shown at $\bar{\phi}_{2a}$ are $\bar{\phi}_{2b}$. At a time angle $\omega t + \delta$ later the vectors shown in Fig. 23 will have moved to the positions shown in Fig. 24. It is important to observe that the oppositely rotating components of the single-phase field are exactly the same in Fig. 21 and in Fig. 24. That is, as far as the reaction produced in the single-phase winding is concerned, there is no difference between positively and negatively rotating fields provided they are always conjugate with respect to the axis of the single-phase winding. In the case of polyphase windings we have seen that positively and negatively rotating magnetic fields produce quite different effects.

A magnetic field that rotates *positively* with respect to a polyphase winding excites the latter so that it in turn produces another field which rotates *positively* and which lags behind the first by an angle determined by the relative generalized angular velocity of the rotating field and the polyphase winding, and the resistance and synchronous self inductance of the latter. The reaction of the polyphase winding is often analyzed in the following manner: The alternating current that is generated in each phase of the polyphase winding produces two component fields that are rotating in opposite directions with respect to the winding. Those component fields due to the different phases which rotate opposite to the exciting field sum up to zero if the phase circuits are symmetrical. The other component fields that rotate in the same direction as the exciting field are in space phase with each other and their resultant is thus n times any one component; where n is the number of phases of the polyphase winding. That is, we may consider that the current in each phase of a polyphase winding consists of two equal components, hereafter designated by the subscripts a and b, each equal to *one-half* of the *maximum phase current.* These two component currents produce equal but oppositely rotating magnetic fields. If the winding were single-phase each of these magnetic fields would exist, but with a symmetrical polyphase winding only the fields exist which rotate in the same direction as does the exciting field. Those that rotate in the opposite direction neutralize one another, so that their resultant is zero.

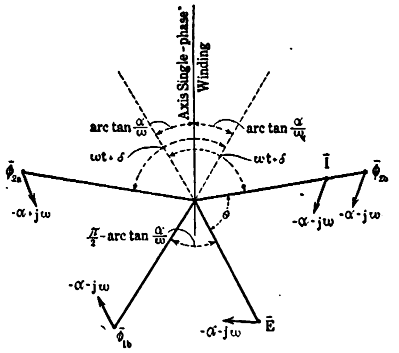

FIG. 24

Let M_1 be the e. m. f. generated in one phase of the polyphase winding by *one ampere* of *either* the a or b component of the single-phase current when varying at the rate of one ampere per second. Similarly let M_n be the e. m. f. generated in the single-phase winding by one ampere of the a component of the polyphase current, provided that is the one which produces a magnetic field, when varying at the rate of one ampere per second. Thus defined the n-phase mutual inductance is n times the single-phase mutual inductance. If the single-phase winding is supplied with an alternating current having an angular velocity of ω and the winding is rotated at this same velocity, ω, with respect to the polyphase winding, one of the oppositely rotating fields due to the single-phase current will be stationary with respect to the polyphase winding and thus will generate no e. m. f. in it. The other field will rotate at a relative angular velocity of $2\,\omega$ and generate an e. m. f., E, in each phase of the polyphase winding. If the single-phase current is I, the single-phase mutual inductance is $M_1 = \dfrac{E}{2\,\omega\, I/2}$. If the single- and polyphase windings are fixed with respect to each other and one phase of the latter is directly opposite the former, the oppositely rotating fields produce equal e. m. fs. in time phase with each other in this phase of the polyphase winding. If E is still the e. m. f. generated in this phase of the polyphase winding the e. m. f. due to *one* of the component fields is $E/2$ and the single-phase mutual inductance is as before $M_1 = \dfrac{E}{2\,\omega\, I/2}$; where I and ω are the current and its angular velocity in the single-phase winding. If the two windings are fixed with respect to each other and balanced currents of I amperes per phase are supplied to the polyphase winding the polyphase mutual inductance is the e. m. f. generated in the single-phase winding divided by $\dfrac{\omega I}{2}$; where ω is the angular velocity of the polyphase currents and I is the current per phase. Thus measured, the n-phase mutual inductance will be n times the single-phase inductance.

Let us assume that we have a three-phase winding fixed in position and a single-phase winding which is rotating in the positive direction at an angular velocity of p radians per second. If the single-phase winding carries a current having an angular velocity of ω radians per second there will be produced in the air-gap two rotating fields, one traveling in the positive direction at an angular velocity of $(\omega + p)$ and the other in the negative direction at a velocity of $(\omega - p)$ radians per second. If p is greater than ω both of these fields travel in the positive direction. The following relations can be worked out on the assumption that ω is either greater or less than p. In accordance with the established custom when analyzing polyphase induction motors and generators, we will choose the first of these assumptions, *viz.*, that ω is greater than p.

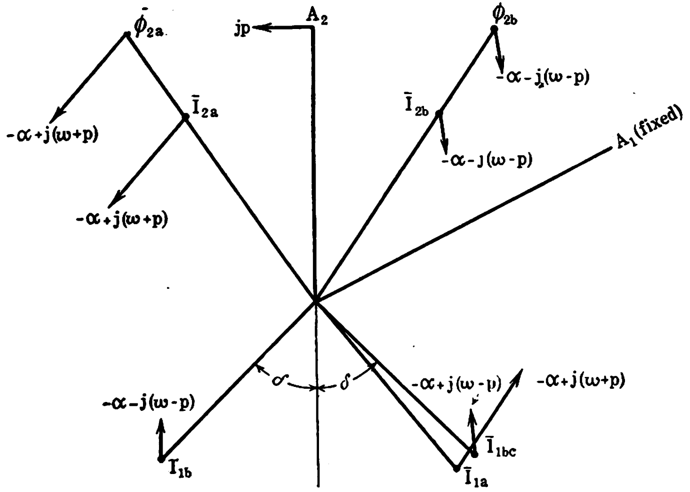
$\bar{\phi}_{2a}$
jp
A_2
$\bar{\phi}_{2b}$
$-\alpha - j(\omega - p)$
$\bar{I}_{2a}$
$\bar{I}_{2b}$
$-\alpha + j(\omega + p)$
$-\alpha - j(\omega - p)$
A_1 (fixed)
$-\alpha + j(\omega + p)$
σ
δ
$-\alpha + j(\omega - p)$
$-\alpha + j(\omega + p)$
$-\alpha - j(\omega - p)$
$\bar{I}_{1bc}$
I_{1b}
$\bar{I}_{1a}$

FIG. 25

The alternating current in the single-phase winding having an angular velocity, ω, produces two oppositely rotating fields each having this same angular velocity. The single-phase winding, having a positive angular velocity of p radians per second, gives to each of these fields an added angular velocity, so that one rotates positively with an angular velocity of $\omega + p$ and the other negatively with angular velocity of $\omega - p$ radians per second. Each of these fields produces by its action on the polyphase winding a rotating field which has the same angular velocity. This is illustrated in Fig. 25. In this figure A_1 represents the fixed axis of one phase of the polyphase winding, and A_2 represents the axis of the single-phase winding; the latter having a positive angular velocity of p radians per second. The oppositely rotating fields due to the single-phase current, which has a complex angular velocity of $- \alpha + j\,\omega$ radians per second, are represented by the vectors $\bar{\phi}_{2a}$ and $\bar{\phi}_{2b}$. The former has an angular velocity of $- \alpha + j\,(\omega + p)$ and the latter a velocity of $- \alpha - j\,(\omega - p)$ radians per second. Since these component fields are proportional to the current in the single-phase winding, this current can be divided into similar components as indicated in the figure, $\bar{I}_{2a}$ and $\bar{I}_{2b}$. Observe that each of these component currents is one-half of the maximum value of the single-phase current, and that at any moment the actual current in the single-phase winding is their vector sum. The first of these rotating fields, $\bar{\phi}_{2a}$, produces a current in the short-circuited polyphase winding $\bar{I}_{1a}$ where

$$I_{1a} = \frac{- \bar{I}_{2a} M_1 (m + j\,p)}{r_1 + (m + j\,p)\,L_1}$$

$$= \frac{- \bar{I}_{2a} M_1 (m + j\,p)}{Z_{1\,(m+jp)}} \qquad (39)$$

M_1 is the single-phase mutual inductance, $m = - \alpha + j\,\omega$ is the generalized angular velocity of the single-phase current, r_1 and L_1 are respectively the resistance and synchronous self inductance of the polyphase winding. The other rotating field $\bar{\phi}_{2b}$ produces a current in the short-circuited polyphase winding, $\bar{I}_{1b}$, which has a *negative* angular velocity of $\omega - p$ radians per second. In this case the current is:

$$\bar{I}_{1b} = \frac{- \bar{I}_{2b} M_1 [- \alpha - j\,(\omega - p)\,]}{r_1 + [- \alpha - j\,(\omega - p)\,]\,L_1} \qquad (40)$$

Each of these polyphase currents produces two oppositely rotating component currents in the single-phase winding. The sum of the components which rotate in the same direction must of course be equal to the initial components rotating in the corresponding direction that were assumed to exist, *viz.*, $\bar{I}_{2a}$ and $\bar{I}_{2b}$. By the principle already established the rotating field due to $\bar{I}_{1b}$, produces the same effect in the single-phase winding as would a current of the same magnitude which rotates positively and is always conjugate to $\bar{I}_{1b}$ with respect to the axis of the single-phase winding. It is readily seen that this conjugate current, $\bar{I}_{1bc}$, is

$$\bar{I}_{1bc} = \frac{- \bar{I}_{2a} M_1 [- \alpha + j\,(\omega - p)\,]}{r_1 + [- \alpha + j\,(\omega - p)\,]\,L_1}$$

$$= \frac{- \bar{I}_{2a} M_1 (m - j\,p)}{Z_{1\,(m-jp)}} \qquad (41)$$

The stator current $\bar{I}_{1a}$ and the conjugate current $\bar{I}_{1bc}$, which replaces $\bar{I}_{1b}$, both rotate positively and their resultant effect on the single-phase winding is thus the same as the effect of their vector sum. The positively rotating component of the single-phase current is thus given by:

$$2\,\bar{I}_{2a} = \frac{- (\bar{I}_{1a} + \bar{I}_{1bc})\,3\,M_1 (- \alpha + j\,\omega)}{r_2 + (- \alpha + j\,\omega)\,L_2}$$

$$\bar{I}_{2a} = \frac{- (\bar{I}_{1a} + \bar{I}_{1bc})\,3/2\,M_1\,m}{Z_{2\,m}} \qquad (42)$$

These characteristic equations may be written thus:

$$\bar{I}_{1a} Z_{1\,(m+jp)} + \bar{I}_{2a} M_1 (m + j\,p) = 0 \qquad (43)$$

$$\bar{I}_{1bc} Z_{1\,(m-jp)} + \bar{I}_{2a} M_1 (m - j\,p) = 0 \qquad (44)$$

$$\bar{I}_{2a} Z_{2\,(m)} + (\bar{I}_{1a} + \bar{I}_{1bc})\,\frac{3\,M_1}{2}\,m = 0 \qquad (45)$$

In these equations it is assumed that any impedance outside the armature and field circuits is negligible, otherwise the zeros at the right of the equality signs would be replaced by the proper *fall* in *terminal pressure*.

If $\bar{I}_{1a}$ and $\bar{I}_{1bc}$ as determined by the first two equations are substituted in the third equation, the latter becomes, on dividing by I_{2a}·

$$Z_{2\,(m)} - 3/2\,M_1\,m \left(\frac{M_1 (m + j\,p)}{Z_{1\,(m+jp)}} + \frac{M_1 (m - j\,p)}{Z_{1\,(m-jp)}} \right) = 0 \qquad (46)$$

If the values of the impedances are substituted, this reduces to the following cubic in m:

$$m^3 + \frac{(k_1\,(1+\sigma) + k_2)\,m^2}{\sigma} + \frac{(k_1^2 + 2\,k_1\,k_2 + \sigma\,p^2)\,m}{\sigma} + \frac{(k_1^2 + p^2)\,k_2}{\sigma} = O \qquad (47)$$

where $k_1 = r_1/L_1$, $k_2 = r_2/L_2$ and $\sigma = 1 - \dfrac{3\,M_1^2}{2\,L_1\,L_2}$

r_1 and r_2 are the resistances of one phase of the polyphase winding and of the single-phase winding; L_1 and L_2 are the synchronous self-inductance of the polyphase winding, and the self inductance of the single-phase winding. At least one root of a cubic equation must be real and since the coefficients are all positive this root must be negative. This indicates that one of the components of the single-phase current is a diminishing direct current. The other roots of the cubic equation are conjugate imaginaries. This indicates two alternating currents, having positive and negative angular velocities but the same rate of diminution. In the polyphase winding, however, the three components of the current will all have different angular velocities. The first component of the single-phase current will produce a current having an angular velocity of p radians per second and the same rate of diminution that it itself has. The second component of the single-phase current, having a positive angular velocity will produce a current which has an angular velocity of approximately $2\,p$ radians per second. The third component of the single-phase current having a negative angular velocity, produces a current in the polyphase winding of very small angular velocity. The sum of the angular velocities of these two latter components of the polyphase current is exactly $2\,p$, and each diminishes at the same rate as does the alternating current in the single-phase winding.

A literal cubic equation is practically impossible to solve but if numerical values are given to k_1, k_2 and σ, the real root can be found to any degree of approximation by Horner's method. The equation may then be reduced to a quadratic, the roots of which can be readily found. Approximate values of the roots may be found as follows: The largest term in any of the coefficients is p^2. With a 60-cycle machine this would be 142,122. The ratios of resistance to self-inductance k_1 and k_2, are of the order ten, while σ is of the order one-tenth. For these values of k_1, k_2 and σ the real root of the characteristic equation is approximately

$$m = -\,k_2/\sigma \qquad (48)$$

The degree of approximation of course depends upon the values of the constants. If $k_1 = k_2 = 1.0$, $\sigma = 0.1$ and $p = 377$;

$$m = -\,10.06$$

whereas the approximate value would be

$$m = 10.0$$

In this case the error is about one-half of one per cent; which is better than the analysis warrants.

Since the sum of the roots must be the coefficient of m^2 in the characteristic equation with its sign changed and since the remaining roots are conjugate imaginaries of the form $-\,\alpha_2 \pm j\,\omega$ we have

$$-\,k_2/\sigma - 2\,\alpha_2 = -\,\frac{k_1\,(1+\sigma) + k_2}{\sigma} \qquad (49)$$

That is:

$$\alpha_2 = -\,\frac{k_1\,(1+\sigma)}{2\,\sigma} \qquad (50)$$

Furthermore since the product of the roots is equal to the last term in the characteristic equation we have:

$$\alpha_1\,(\alpha_2^2 + \omega^2) = (k_1^2 + p^2)\,k_2/\sigma$$

From this, the approximate value of ω is seen to be

$$\omega = \sqrt{p^2 - (\alpha_2^2 - k_1^2)}$$

$$= p - \frac{\alpha_2^2 - k_1^2}{2\,p} \qquad \text{approximately} \quad (51)$$

For the assumed values of k_1, k_2, σ and p the calculated values of α_2 are ω are:

$$\alpha_2 = 5.497$$

and $\omega = 376.73$

For the same values of the constants the approximate values of α_2 and ω are:

$$\alpha_2 = \frac{1 \times 1.12}{2 \times 0.1} = 5.50$$

$$\omega = 377 - \frac{5.5^2 - 1}{2 \times 377}$$

$$= 376.96$$

It is readily appreciated that for such values of the constants as were chosen the roots are approximately

$$m' = -\,k_2/\sigma \qquad (52)$$

$$m'' = -\,\frac{k_1\,(1+\sigma)}{2\,\sigma} \pm j\,p \qquad (53)$$

It may be desirable, however, to extend the analysis so that it will apply to the case in which it is necessary to consider the constants of circuits outside of the machine itself. In that event the approximate values of the roots may not be sufficiently accurate. Again if the constants are considerably larger, as for example:

Let $k_1 = 15$, $k_2 = 20$, $\sigma = 0.15$ and $p = 377$ the actual values of m are:

$$m' = -\,143.2 + j\,o$$
$$m'' = -\,52.57 \pm j\,360.3$$

The approximate values of m are:

$$m' = -\,133.3$$
$$m'' = -\,57.5 \pm j\,373$$

In this case the error in the approximate calculation is quite considerable.

The general principle that determines the magnitude of the various component currents is that the resulting distribution of magnetic field in all parts of the machine cannot abruptly change, and what is even more important the component fields due to the different currents cannot abruptly change. That is, in mathematical language, the magnetic field at every point is a continuous function of the time, and since the magnetic fields are assumed to be proportional to the currents, the latter are also continuous functions of the time. If it were possible to construct mutually inductive circuits which had a leakage coefficient of zero, the currents in these circuits might abruptly change, *i. e.*, they might be discontinuous functions.

In order to illustrate the method of determining the component currents we will calculate their approximate values when the alternator is suddenly short-circuited. The general method of treating the single-phase current is to represent it by pairs of oppositely rotating vectors. In the present problem there are two such pairs one having no velocity and the other an angular velocity of p radians per second. That is, the transient current $\bar{I}_{2T}$ in the single-phase winding may be equated to four components:

$$\bar{I}_{2T} = \bar{I}_{2a}' + \bar{I}_{2b}' + \bar{I}_{2a}'' + \bar{I}_{2b}'' \tag{54}$$

$\bar{I}_{2a}'$ and $\bar{I}_{2b}'$ are conjugate and have zero velocity with respect to the single-phase axis. $\bar{I}_{2a}''$ and $\bar{I}_{2b}''$ are conjugate and have an angular velocity of approximately p radians per second with respect to the same axis. Each of these transient components in the single-phase winding is the cause of a corresponding component in the polyphase winding. Thus, similarly:

$$\bar{I}_{1T} = \bar{I}_{1a}' + \bar{I}_{2b}' + \bar{I}_{1a}'' + \bar{I}_{1b}'' \tag{55}$$

$\bar{I}_{1a}'$ and $\bar{I}_{1b}'$ both have the same angular velocity of p radians per second, $\bar{I}_{1a}''$ has an angular velocity of approximately $2\,p$ and I_{1b}'' an angular velocity of approximately zero radian per second.

The fundamental relation at the moment of short circuit is that:

$$I_{10} = \bar{I}_{1T} + I_{1s} \tag{56}$$

and

$$I_{20} = \bar{I}_{2T} + I_{2s} \tag{57}$$

where I_{10} is the current in the polyphase winding, before short circuit and I_{1s} is the steady current after the transient $\bar{I}_{1T}$ occasioned by the short circuit has disappeared. I_{20} and I_{2s} are similar values for the single-phase winding.

If the field excitation is not changed at or after short circuit $I_{20} = I_{2s}$ and thus the transient current in the single-phase winding must be zero at the moment of short circuit. Since the components of the single-phase current are conjugate in pairs it follows that at the moment of short circuit:

$$I_{2a}' = -\, I_{2a}'' \tag{58}$$

and

$$I_{2b}' = -\, I_{2b}'' \tag{59}$$

The corresponding pairs of single- and polyphase component currents must satisfy the two equations (**43**) and (**44**), the *a* components satisfying (**43**) and the *b* components, (**44**). Thus:

$$I_{1a}'\,[r_1 + (-\,k_2/\sigma + j\,p)\,L_1] + I_{2a}'\,M_1\,(-\,k_2/\sigma + j\,p) = O \tag{60}$$

$$I_{1b}'\,[r_1 + (-\,k_2/\sigma + j\,p)\,L_1] + I_{2b}'\,M_1\,(-\,k_2/\sigma + j\,p) = O \tag{61}$$

$$I_{1a}''\left[r_1 + \left(-\,k_1\,\frac{(1+\sigma)}{2\,\sigma} + j\,2\,p\right)L_1\right] + I_{2a}''\,M_1\left(-\,k_1\,\frac{(1+\sigma)}{2\,\sigma} + j\,2\,p\right) = O \tag{62}$$

$$I_{1b}''\left(r_1 + (-\,k_1\,\frac{(1+\sigma)}{2\,\sigma}\,L_1\right) + I_{2b}''\,M_1\left(-\,k_1\,\frac{(1+\sigma)}{2\,\sigma}\right) = O \tag{63}$$

If each of these equations is divided by L_1, it will be seen that since k_1 is much smaller than k_1/σ, k_2/σ or p, the following relations between the polyphase and single-phase components are approximately true:

$$\bar{I}_{1a}' = -\,M_1/L_1\,\bar{I}_{2a}'$$
$$\bar{I}_{1b}' = -\,M_1/L_1\,\bar{I}_{2b}'$$
$$\bar{I}_{1a}'' = -\,M_1/L_1\,\bar{I}_{2a}''$$

but

$$\bar{I}_{1b}'' = -\,M_1/L_1\,\frac{(1+\sigma)}{(1-\sigma)}\,\bar{I}_{2b}''$$

If the conditions set forth in equations (**55**), (**58**) and (**59**) are applied to these four equations we have:

$$\bar{I}_{1T} = M_1/L_1\,\frac{2\,\sigma}{1-\sigma}\,I_{2b}' \tag{64}$$

If it is deemed necessary a more accurate value of I_{2b}' can be calculated by substituting in equations (**60**) to (**63**) inclusive the actual values of the generalized angular velocity as determined by equation (**47**) and by recognizing that I_{2a}' and I_{2b}' are conjugate with respect to the single-phase axis.

Since the current in the polyphase winding before short circuit is much smaller than the steady short-circuit current:

$\bar{I}_{1T} = -\,I_{1s}$ approximately, at the moment of short circuit. The steady short-circuit current is calculated by dividing the e. m. f. generated in the polyphase winding on open circuit by the synchronous impedance. The open circuit e. m. f. is

$$E = -\,j\,p\,M_1\,I_{20} \tag{65}$$

Since the resistance is much less than the synchronous reactance the steady short-circuit current is approximately.

$$I_{1s} = -\,\frac{j\,p\,M_1\,I_{20}}{j\,p\,L_1} \tag{66}$$

$$= -\,M_1/L_1\,I_{20}$$

From this it follows that:

$$I_{2b}' = I_{20}\,\frac{(1-\sigma)}{2\,\sigma} \tag{67}$$

The direct-current component of the single-phase current has an initial value of $I_{2a}' + I_{2b}'$ which is $I_{20} \dfrac{(1-\sigma)}{\sigma}$. The alternating component of the single-phase current is equal and opposite to this initially, but one-half cycle later the vectors representing all of the component currents are in phase. During this time, however, the direct and alternating components have shrunk different amounts. The first maximum value of the single-phase current will be, approximately:

$$I_{20} \frac{(1-\sigma)}{\sigma} \left\{ \epsilon^{-\frac{k_2}{\sigma}\frac{\pi}{p}} + \epsilon^{-\frac{k_1(1+\sigma)}{2\sigma}\frac{\pi}{p}} \right\} \tag{68}$$

For $k_1 = k_2 = 1$, $\sigma = 0.1$, this is $17\, I_{20}$.

The component of the polyphase current which has an angular velocity of p radians per second is $\bar{I}_{1a}' + \bar{I}_{1b}'$ which initially equals $\dfrac{1-\sigma}{\sigma} I_{1s}$. The component which has an angular velocity of approximately $2\,p$ is $\bar{I}_{1a}''$ which initially equals $-\dfrac{1-\sigma}{2\,\sigma} I_{1s}$. The component which has a very small angular velocity is I_{1b}'' which initially equals $-\dfrac{1+\sigma}{2\,\sigma} I_{1s}$. The first of these components shrinks at a rate of k_2/σ while each of the others shrinks at a rate of $\dfrac{k_1(1+\sigma)}{2\,\sigma}$. Notice that initially the first component is opposite in phase to each of the others. Thus one-half cycle after the transient begins, the three components will be in phase. If the transient begins at such a time that the steady short-circuit current in one phase would naturally be passing through its maximum value, this phase will carry the maximum possible transient current one-half cycle later. This maximum transient current is approximately:

$$I_{1s} \left[\frac{1-\sigma}{\sigma} \epsilon^{-\frac{k_2}{\sigma}\frac{\pi}{p}} + \frac{1}{\sigma} \epsilon^{-k_1 \frac{(1+\sigma)}{2\sigma}\frac{\pi}{p}} \right] \tag{69}$$

for $k_1 = k_2 = 1$, $= 0.1$ this is $18\, I_{1s}$, *i. e.* eighteen times the maximum value of the steady short circuit current.

TWO SINGLE-PHASE WINDINGS HAVING A CONSTANT RELATIVE ANGULAR VELOCITY

A current of angular velocity ω in one single-phase winding produces in the other winding currents having angular velocities of $\omega + p$ and $\omega - p$; where p is the relative angular velocity of the two windings. Each of these currents reacts on the first winding and produces additional currents having angular velocities of $\omega + 2\,p$ and $\omega - 2\,p$. These currents in turn produce others in the second winding having angular velocities of $\omega + 3\,p$ and $\omega - 3\,p$, and so on ad infinitum. It is then evident that the solution of this problem cannot be reached by the method here developed unless all but a few of the component currents are neglected. The solution will not be attempted at this time.

Carson[5] has suggested a functional form of solution of this problem and Dreyfus[6] also gives a solution based on certain assumptions.

In conclusion the writer hardly needs to say that it has been his sole purpose to show how certain low-frequency transient conditions in electric machinery may be analyzed by a vector method. Before accurate results can be obtained by this method considerable experimental work is probably necessary in order to determine the most satisfactory methods of measuring the required constants. An interesting development would be the solution of the transient conditions in certain unsymmetrical arrangements of polyphase machines similar to those so ably discussed by Mr. Fortescue.[7]

Transient conditions may be the result of a variety of causes, one of which is short circuit. The vector method, however, is equally applicable in any case in which the original premises are reasonably true. The actual working out of a numerical solution is rather complex and this has been purposely avoided in order that the general principles might not be obscured a mass of calculation. Furthermore, it seemed best at this time not to attempt to generalize the theory and present it in such form that it would apply to combinations of electric machines that might even be unsymmetrical.

Appendix

For those who are interested in comparing the vector method of solution with the differential equations which apply to these problems the following development is given.

Consider a symmetrically wound three-phase induction motor having Y-connected stator and rotor phases as shown diagrammatically in Fig. 26. The stator and rotor currents are represented by the letters x and y respectively and their assumed positive directions are indicated by the arrows. Fig. 27 shows a development of the air gap. The progression of the phases, 1, 2 and 3 is toward the right for both stator and rotor, and the latter moves at a constant angular velocity of p radians per second toward the right.

Let r_1, r_2, L'_1 and L_2' represent the resistances and self-inductances of one phase of stator and rotor respectively.

Let M_x represent the mutual inductance between any

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5. Bibliography No. 7.
6. Bibliography No. 4.
7. A. I. E. E. Vol. XXXVII, p. 1027.

two phases of the stator. Due to symmetry these three mutual inductances are equal.

Let M_y represent the mutual inductances between any two phases of the rotor. These mutual inductances are likewise equal.

Let $2/3\,M$ represent the maximum value of the mutual inductance between one phase of the stator and one phase of the rotor. At a time when the angular displacement of these phases of stator and rotor is $p\,t$ the momentary value of this mutual inductance is $2/3\,M \cos p\,t$, if the space distribution of the air-gap flux density is sinusoidal. This assumption of a sinusoidal distribution of flux density is not much in error if the motor is well designed. It is also necessary to assume that r_1, r_2, L_1', L_2', M_x, M_y and $2/3\,M$ are constant. This is commonly done in engineering practise.

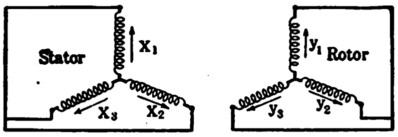

FIG. 26

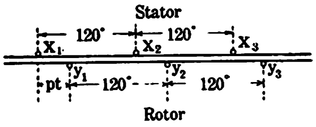

FIG. 27

With the stator phases short-circuited as indicated in Fig. 26 the following differential equations may be written,—considering first phases 1 and 2 and then phases 2 and 3.

$$\Big\{ r_1 x_1 + L_1' \frac{d\,x_1}{d\,t} - M_x \frac{d\,x_2}{d\,t} - M_x \frac{d\,x_3}{d\,t}$$

$$+ \frac{d}{d\,t} \Big[y_1 \frac{2}{3} M \cos p\,t + y_2 \frac{2}{3} M \cos (p\,t + 120^\circ)$$

$$+ y_3 \frac{2}{3} M \cos (p\,t + 240\ ) \Big] \Big\}$$

$$- \Big\{ r_1 x_2 + L_1' \frac{d\,x_2}{d\,t} - M_x \frac{d\,x_3}{d\,t} - M_x \frac{d\,x_1}{d\,t}$$

$$+ \frac{d}{d\,t} \Big(y_2 \frac{2}{3} M \cos p\,t + y_3 \frac{2}{3} M \cos (p\,t + 120^\circ)$$

$$+ y_1 \frac{2}{3} M \cos (p\,t + 240^\circ) \Big\} = 0 \qquad (1)$$

There is a negative sign before M_x for the reason that positive currents in phases 2 and 3, for example, produce a flux through phase 1 which is opposite in direction to the flux that phase 1 produces through itself.

The equation which applies to phases 2 and 3 is:

$$\Big\{ r_1 x_2 + L_1' \frac{d\,x_2}{d\,t} - M_x \frac{d\,x_3}{d} - M_x \frac{d\,x_1}{d\,t}$$

$$+ \frac{d}{d\,t} \Big[y_2 \frac{2}{3} M \cos p\,t + y_3 \frac{2}{3} M \cos (p\,t + 120^\circ)$$

$$+ y_1 \frac{2}{3} M \cos (p\,t + 240\ ) \Big] \Big\}$$

$$- \Big\{ r_1 x_3 + L_1' \frac{d\,x_3}{d\,t} - M_x \frac{d\,x_1}{d\,t} - M_x \frac{d\,x_2}{d\,t}$$

$$+ \frac{d}{d\,t} \Big[y_3 \frac{2}{3} M \cos p\,t + y_1 \frac{2}{3} M \cos (p\,t + 120^\circ)$$

$$+ y_2 \frac{2}{3} M \cos (p\,t + 240^\circ) \Big] \Big\} = 0 \quad (2)$$

Since the sums of the stator and rotor currents are each equal to zero, we may substitute:

$$x_3 = - x_1 - x_2 \quad \text{and} \quad y_5 = - y_1 - y_2$$

Also replace $L_1' + M_x$ by L_1. Multiply the second equation by 2 and add it to the first; again, subtract the second equation from the first. The results of these operations are respectively:

$$r_1 x_1 + L_1 \frac{d\,x_1}{d\,t} + \frac{2}{\sqrt{3}} M \frac{d}{d\,t} \Big[y_1 \cos (p\,t + 30\ )$$

$$+ y_2 \cos (p\,t + 90^\circ) \Big] = 0 \qquad (3)$$

$$r_1 x_2 + L_1 \frac{d\,x_2}{d\,t} + \frac{2}{\sqrt{3}} M \frac{d}{d\,t} \Big[y_1 \cos (p\,t - 90\ )$$

$$+ y_2 \cos (p\,t - 30^\circ) \Big] = 0 \qquad (4)$$

By a similar procedure differential equations may be written for phases 1 and 2 of the rotor. These equations will be found to be similar to (3) and (4) except that the signs before the phase angles will be reversed. Thus we will find that:

$$r_2 y_1 + L_2 \frac{d\,y_1}{d\,t} + \frac{2}{\sqrt{3}} M \frac{d}{d\,t} \Big[x_1 \cos (p\,t - 30^\circ)$$

$$+ x_2 \cos (p\,t - 90^\circ) \Big] = 0 \qquad (5)$$

$$r_2 y_2 + L_2 \frac{d\,y_2}{d\,t} + \frac{2}{\sqrt{3}} M \frac{d}{d\,t} \Big[x_1 \cos (p\,t + 90^\circ)$$

$$+ x_2 \cos (p\,t + 30^\circ) \Big] = 0 \qquad (6)$$

The following treatment was suggested to the author by Dr. F. L. Hitchcock as being the simplest method of reducing these equations to one containing a single variable.

Let $\quad r_1 x_1 + L_1 \dfrac{d\,x_1}{d\,t} = Z_1 x_1$

(Z_1 is an impedance operator)

Similarly $\quad r_1 x_2 + L_1 \dfrac{d\,x_2}{d\,t} = Z_1 x_2$

Also $\quad r_2 y_1 + L_2 \dfrac{d\,y_1}{d\,t} = Z_2 y_1$

And $\quad r_2 y_2 + L_2 \dfrac{d\,y_2}{d\,t} = Z_2 y_2$

If D and D^{-1} represent the direct and inverse differential operators, we shall have on integrating equations (3) and (4):

$$D^{-1} Z_1 x_1 + 2/\sqrt{3}\, M\, [y_1 \cos (p\,t + 30^\circ)$$

$$+ y_2 \cos (p\,t + 90^\circ] = 0 \qquad (7)$$

$$D^{-1} Z_1 x_2 + 2/\sqrt{3}\, M\, [y_1 \cos (p\,t - 90^\circ)$$

$$+ y_2 \cos (p\,t - 30^\circ] = 0 \qquad (8)$$

Solve these two equations for y_1 and y_2

$$y_1 = \frac{2}{\sqrt{3M}} \{\cos (p t + 90°) D^{-1} Z_1 x_2$$

$$- \cos (p t - 30\ ) D^{-1} Z_1 x_1\} \qquad (9)$$

$$y_2 = \frac{2}{\sqrt{3M}} \{\cos (p t - 90\ ) D^{-1} Z_1 x_1$$

$$- \cos (p t + 30°) D^{-1} Z_1 x_2\} \qquad (10)$$

Differentiation of equation (**9**) gives:

$$\frac{d y_1}{d t} = \frac{2}{\sqrt{3M}} \{\cos (p t + 90°) Z_1 x_2$$

$$- p \sin (p t + 90°) D^{-1} Z_1 x_2 - \cos (p t - 30°) Z_1 x_1$$

$$\mp p \sin (p t - 30°) D^{-1} Z_1 x_1\}$$

The expression for y_2 may be differentiated in a like manner.

If the values of y_1, y_2, $\frac{d y_1}{d t}$ and $\frac{d y_2}{d t}$ are now substituted in equations (**5**) and (**6**), the resulting equations may be written as follows:

In order to simplify the writing of these equations let:

$$\cos (p t - 30°) = a \qquad \sin (p t - 30°) = a_1$$
$$\cos (p t + 30°) = b \qquad \sin (p t + 30°) = b_1$$
$$\cos (p t - 90°) = c \qquad \sin (p t - 90°) = c_1$$
$$\cos (p t + 90°) = d \qquad \sin (p t + 90°) = d_1$$

Equation (**5**) becomes:

$$\{ (- r_2 a + p L_2 a_1) D^{-1} Z_1 - L_2 a Z_1$$
$$+ M^2 (a D - p a_1) \} x_1$$
$$+ \{ (r_2 d - p L_2 d_1) D^{-1} Z_1 + L_2 d$$
$$+ M^2 (c D - p c_1) \} x_2 = 0 \qquad (11)$$

Equation (**6**) becomes:

$$\{ (r_2 c - L_2 p c_1) D^{-1} Z_1 + L_2 c Z_1$$
$$(- r_2 b + L_2 p b_1) D^{-1} Z_1 - L_2 b Z_1$$
$$+ M^2 (d D - p d_1) \} x_1 +$$
$$+ M^2 (b D - p b_1) \} x_2 = 0 \qquad (12)$$

In the process of eliminating x_1 and x_2 from equations (**11**) and (**12**) all of the harmonic terms in the coefficients vanish, giving as a final result:

$$(r_2^2 + L_2^2 p^2) (D^{-1} Z_1)^2 + L_2^2 Z_1^2 + (D^2 + p^2) M^4$$
$$+ 2 r_2 L_2 Z_1 D^{-1} Z_1 - 2 (r_2 D + p^2 L_2) M^2 D^{-1} Z_1$$
$$- 2 M^2 L_2 Z_1 D = 0 \qquad (13)$$

Differentiate this equation twice, collect the terms in descending powers of D and after dividing each term by $L_1^2 L_2^2$, let $1 - \frac{M^2}{L_1 L_2} = \sigma$, $r_1/L_1 = k_1$ and $r_2/L_2 = k_2$. This gives:

$$\sigma^2 D^4 + 2 \sigma (k_1 + k_2) D^3 + [(k_1 + k_2)^2 + 2 \sigma k_1 k_2$$
$$+ p^2 \sigma^2] D^2 + [2 k_1 k_2 (k_1 + k_2) + 2 p^2 k_1 \sigma] D$$
$$+ k_1^2 (k_2^2 + p^2) = 0 \qquad (14)$$

If equation (**26**) in the body of the paper is written for the conjugate roots, *i. e.*, for a negative value of the rotor angular velocity it would be:

$$\sigma m^2 + (k_1 + k_2 + j p \sigma) m + k_1 k_2 + j p k_1 = 0 \qquad (26_1)$$

The product of equations (**26**) and (**26₁**) will contain all four roots, and will be found to be identical with the differential equation (**14**).

This proves that the two methods are identical in regard to the results they produce, and that, of the two, the vector method is much less laborious. Furthermore, it is much simpler to determine the constants of integration by the vector method.

SYNCHRONOUS ALTERNATOR

In this case there is only one phase on the rotor, *viz.* the field winding. Making the same assumptions in regard to the resistances and inductances as were made before, the following differential equations may be written:

$$r_1 x_1 + L_1 \frac{d x_1}{d t} + M \frac{d}{d t} (y \cos p t) = 0$$

$$r_1 x_2 + L_1 \frac{d x_2}{d t} + M \frac{d}{d t} [y \cos (p t - 120)] = 0$$

$$r_2 y + L_2 \frac{d y}{d t} + M \frac{d}{d t} [x_1 \cos (p t - 30°)$$
$$+ x_2 \cos (p t - 90°)] = 0$$

If these equations are treated by the method already outlined the resulting differential equation is:

$$\sigma D^3 + (k_1 + k_2 + \sigma k_1) D^2 + (k_1^2 + 2 k_1 k_2$$
$$+ \sigma p^2) D + k_2 (k_1^2 + p^2) = 0$$

In this case $\sigma = 1 - 3/2 \frac{M^2}{L_1 L_2}$

Again the equation obtained by the vector method and the differential equation are identical.

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